Infinite Series – Sequences

An infinite series is the addition of infinite many quantities. With this, the study of infinite series naturally relies on the notion of a limit, and in this sense is considered another branch of calculus. The are many important and interesting application of infinite series. For example, the value π can be expressed as an infinite series as shown below.

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \ldots$$

Another very important example is the Fourier Series, which has innumerable practical applications. It can be shown that any periodic function can be expressed as a combination of an infinite number of sinusoids. The expression, known as a Fourier Series can be written as follows:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \, cos(nx) + b_n \, sin(nx)]$$

Where, a_n and b_n are determined from integral formulas not shown here.

To understand infinite series, we first need to understand sequences, which are simply a list of quantities. For example, the terms of the infinite series representing π from above can be considered a sequence, a_n, where

$$a_n = \frac{4}{1}, \frac{4}{3}, \frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \ldots$$

Therefore, we start our study of infinite series by looking first at sequences.

We start by giving a formal definition of a sequence below.

Sequence
A sequence, $\{a_n\}$, is an ordered collection of numbers that may or may not be defined by a function, f, on a set of sequential integers. The values a_n are called the terms of the sequence, and n is called the index. $$\{a_n\} = a_1, a_2, a_3, \ldots$$ If the sequence is defined by a function, we can say that $$a_n = f(n)$$ Note: The sequence does not have to start at $n = 1$. It can start at any other integer.

A sequence can be generated by an explicit function, recursively, or by no specific formula at all. For example, the following sequence

$$3, 1, 4, 1, 5, 9, 2, 6, \ldots$$

is simply a list of the digits of π, for which there is no specific formula to generate the n^{th} term.

For a recursive sequence we are given the first one or more terms, and then the n^{th} term is computed in terms of the preceding terms using a specific formula. A beautiful example of a recursive sequence is the Fibonacci sequence. The sequence has many applications in science and engineering and it also appears frequently in nature, particularly in biological settings. For example, the number of spiral arms in a sunflower almost always turn out to be a number from the Fibonacci sequence. Let's look at two examples of a recursive sequence, starting with the Fibonacci sequence.

Example 1: The Fibonacci sequence is defined recursively so that each new term is computed as the sum of the previous two terms.

$$F_n = F_{n-1} + F_{n-2}, \quad for\; n > 2$$

Where, $F_1 = F_2 = 1$

Compute F_3, F_4, F_5, F_6, F_7

Solution: Using the recursive formula we have

$$\begin{aligned} F_3 &= F_2 + F_1 = 1 + 1 = 2 \\ F_4 &= F_3 + F_2 = 2 + 1 = 3 \\ F_5 &= F_4 + F_3 = 3 + 2 = 5 \\ F_6 &= F_5 + F_4 = 5 + 3 = 8 \\ F_7 &= F_6 + F_5 = 8 + 5 = 13 \end{aligned}$$

The sequence is then

$$F_n = \{1, 1, 2, 3, 5, 8, 13, \dots\}$$

Example 2: Compute a_2, a_3, a_4 for the sequence defined recursively as follows:

$$a_1 = 1 \qquad\qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

Solution:

$$\begin{aligned} a_2 &= \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5 \\ a_3 &= \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \cong 1.4167 \\ a_4 &= \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408} \cong 1.414216 \end{aligned}$$

Note: As $n \to \infty$ the value of this sequence approaches $\sqrt{2}$.

Lastly, and the type of sequence we will most use, is one that is defined by an explicit function, i.e. $a_n = f(n)$. Let's look at an example.

Example 3: Compute $f(n)$ for $n = 0,1,2,3$ for the following sequences.

$$a_n = f(n) = \frac{1}{2^n} \qquad\qquad a_n = f(n) = \frac{5n-1}{12n+9}$$

Solution:

$$f(0) = \frac{1}{2^0} = 1 \qquad\qquad f(0) = \frac{5\cdot 0-1}{12\cdot 0+9} = -\frac{1}{9}$$

$$f(1) = \frac{1}{2^1} = \frac{1}{2} \qquad\qquad f(1) = \frac{5\cdot 1-1}{12\cdot 1+9} = \frac{5}{21}$$

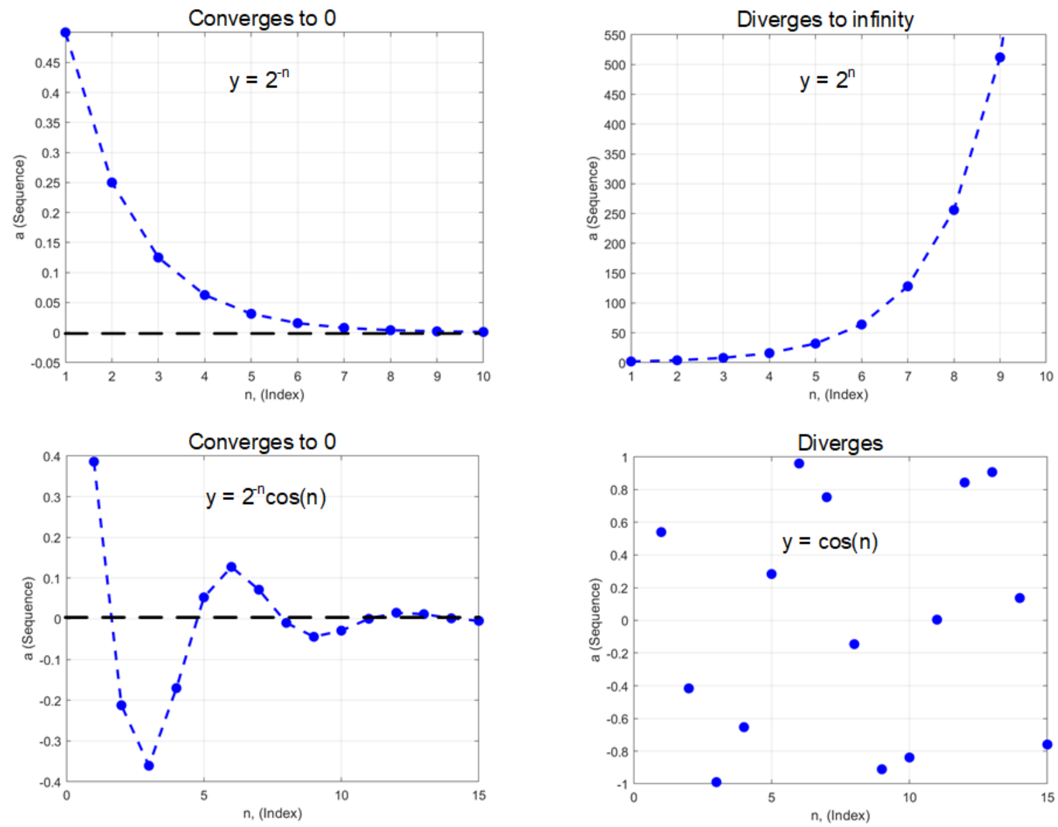
$$f(2) = \frac{1}{2^2} = \frac{1}{4} \qquad\qquad f(2) = \frac{5\cdot 2-1}{12\cdot 2+9} = \frac{9}{33}$$

$$f(3) = \frac{1}{2^3} = \frac{1}{8} \qquad\qquad f(3) = \frac{5\cdot 3-1}{12\cdot 3+9} = \frac{14}{45}$$

One of the main goals in studying sequences, and ultimately infinite series, is to determine convergence. Informally, a sequence converges if the terms of the sequence tend to a particular value, e.g. L, as n gets very large. A formal definition is given below.

Limit of a Sequence
We say that the sequence, $\{a_n\}$, converges to a limit L, and we write $$\lim_{n\to\infty} a_n = L$$ if, for every $\varepsilon > 0$, there is a number M such that $\lvert a_n - L\rvert < \epsilon$ for all $n > M$. • If no limit exists, we say that $\{a_n\}$ diverges. • If the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.

The figures below show various plots of sequences to help visualize convergence and divergence cases. The plots on the left show two different ways a series may converge. The plots on the right show series that diverge. The top one diverges because the terms go to infinity. The bottom plot, while it does not grow unbounded, also diverges in the sense that a limit does not exist.



Let's use the formal definition from above to prove the convergence of a particular sequence.

Example 4: Prove formally that $\lim_{n\to\infty} a_n = 1$ for the following sequence

$$a_n = \frac{n+1}{n+5}$$

Solution: The definition requires the following

$$|a_n - 1| < \epsilon \qquad \text{for all } n > M$$

In our case we have

$$\left|\frac{n+4}{n+1} - 1\right| = \left|\frac{n+4}{n+1} + \frac{-n-1}{n+1}\right| = \left|\frac{3}{n+1}\right| = \frac{3}{n+1}$$

Where, we were able to drop the absolute value since the $n \geq 0$.

Therefore, we now have

$$\frac{3}{n+1} < \epsilon \qquad \textit{or} \qquad n > \frac{3}{\epsilon} - 1$$

In other words, $|a_n - 1| < \epsilon$ if $n > \frac{3}{\epsilon} - 1$, which implies $M = \frac{3}{\epsilon} - 1$ in order for $\lim_{n\to\infty} a_n = 1$

As you can surmise from the above example, using the formal definition for convergence can be quite challenging. Fortunately, we can more easily use techniques involving the limit for sequences that can be defined by function, i.e. $a_n = f(n)$. Furthermore, we can assume n is continuous by letting $n = x$. With that we can say

Given $$a_n = f(n) = \frac{n+4}{n+1}$$ We can define $$f(x) = \frac{x+4}{x+1}$$

We can then use the next theorem, which will allow us to use limit techniques we have developed from calculus 1 to analyze the converge of sequences.

Sequence Defined by a Function
If $\lim\limits_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit: $$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

The following example uses the above theorem to determine the convergence.

Example 5: Determine is the given sequence converges and if it does find the limiting value.

$$a_n = \frac{n^2-2}{n^2}$$

Solution: We start by defining the following continuous function

$$f(x) = \frac{x^2-2}{x^2}$$

Next, we find the limit of the function and use the results to determine the convergence of the original sequence.

$$\begin{aligned} \lim_{x\to\infty}\left(\frac{x^2-2}{x^2}\right) &= \lim_{x\to\infty}\left(1-\frac{2}{x^2}\right) \\ &= \lim_{x\to\infty}(1) - 2\lim_{x\to\infty}\left(\frac{1}{x^2}\right) \\ &= 1 - 2\cdot 0 = 1 \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}(a_n) = \lim_{x\to\infty}\big(f(x)\big) = 1$$

Note: To solve the above limit problem, we used some of the basic limit laws and techniques from calculus 1. You may want to return to the limit sections in calculus 1 to refresh your memory.

Another useful theorem for evaluating the convergence of sequences is stated below.

A Function of a Sequence
If f is continuous and $\lim_{n\to\infty}(a_n) = L$, then $$\lim_{n\to\infty}(a_n) = f\left(\lim_{n\to\infty}(a_n)\right) = f(L)$$

Example 5: Determine if the given sequence converges and if it does find the limiting value.

$$a_n = e^{\left(\frac{n^2-2}{n^2}\right)}$$

Solution: We use the above theorem along with the results from the previous example.

$$\lim_{n\to\infty}\left(e^{\left(\frac{n^2-2}{n^2}\right)}\right) = e^{\left(\lim_{n\to\infty}\left(\frac{n^2-2}{n^2}\right)\right)} = e^{(1)} = e$$

To understand convergence, it's important to understand the concepts of bounded sequences and monotonic sequences. The formal definitions are given below.

Bounded Sequences
A sequence $\{a_n\}$ is: • **Bounded from above** if there is a number M_u such that $a_n \leq M_u$ for all n. The number M_u is called the upper bound.   • **Bounded from below** if there is a number M_d such that $a_n \geq M_d$ for all n. The number M_d is called the lower bound.   The sequence $\{a_n\}$ is called **bounded** if it is bounded from above **and** below. A sequence that is not bounded is called an **unbounded** sequence.

Monotonic Sequences
A sequence $\{a_n\}$ is **monotonic** if: $a_{n+1} > a_n$, i.e. it is increasing.   or $a_{n+1} < a_n$, , i.e. it is decreasing.

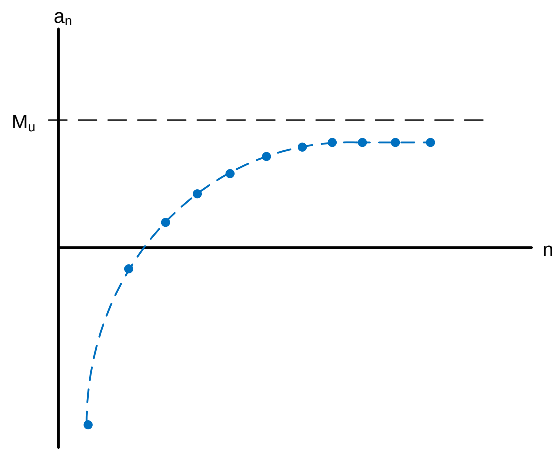
Let's now see how these two definitions can be used to better understand convergence. We start by combining the concept of a sequence being bounded from above or below with the concept of monotonic sequences.

Monotonic Sequences Bounded from Above or Below

If a sequence $\{a_n\}$ is:

- **Monotonically increasing,** $a_{n+1} > a_n$, and
- **Bounded from above,** $a_n \leq M_u$

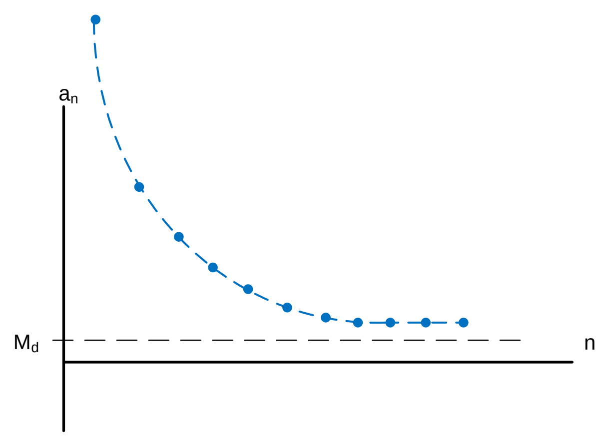
Then $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n) \leq M_u$



If a sequence $\{a_n\}$ is:

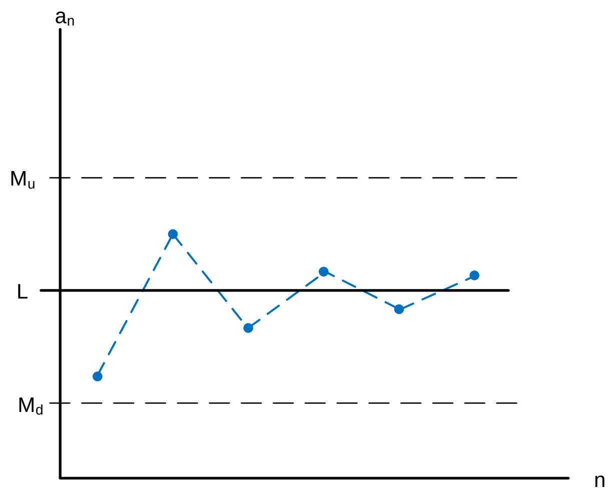
- **Monotonically decreasing,** $a_{n+1} < a_n$, and
- **Bounded from below,** $a_n \geq M_d$

Then $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n) \geq M_d$



Finally, we state a theorem related to sequences that are bounded from above and below.

Convergent Sequences are Bounded
If $\{a_n\}$ converges, then it is bounded from above and below.



Note: This theorem does **not** state that all bounded sequences converge, but rather that all converging sequences are bounded.

Finally, let's do some examples to practice what we have learned up to this point.

Example 6: Compute the first four terms of the given sequences.

$$a. \quad a_n = (-1)^{n+1} \qquad\qquad b. \quad a_0 = 1 \qquad a_n = a_{n-1} + \frac{1}{a_{n-1}}$$

Solution:

a.

$$a_0 = (-1)^{0+1} = (-1)^1 = -1$$
$$a_1 = (-1)^{1+1} = (-1)^2 = 1$$
$$a_2 = (-1)^{2+1} = (-1)^3 = -1$$
$$a_3 = (-1)^{3+1} = (-1)^4 = 1$$

As this sequence oscillates between one and negative one it does not converge.

b. In this case the sequence is defined recursively.

$$a_0 = 1$$

$$a_1 = a_0 + \frac{1}{a_0} = 1 + \frac{1}{1} = 2$$

$$a_2 = a_1 + \frac{1}{a_1} = 2 + \frac{1}{2} = 2.5$$

$$a_3 = a_2 + \frac{1}{a_2} = 2.5 + \frac{1}{2.5} = 2.9$$

Example 7: Determine the limit of the given sequences or state that the sequence diverges.

a. $a_n = \sqrt{4 + \frac{1}{n}}$

b. $a_n = e^{\left(\frac{4n}{3n+9}\right)}$

c. $a_n = cos\left(\frac{n^2}{2n^4+3n+4}\right)$

d. $a_n = tan^{-1}(e^{-n} + 1)$

Solution:

a.

$$\lim_{n\to\infty}\left(\sqrt{4 + \frac{1}{n}}\right) = \sqrt{\lim_{n\to\infty}\left(4 + \frac{1}{n}\right)}$$

$$= \sqrt{\lim_{x\to\infty}\left(4 + \frac{1}{x}\right)}$$

$$= \sqrt{\lim_{x\to\infty}(4) + \lim_{x\to\infty}\left(\frac{1}{x}\right)}$$

$$= \sqrt{4 + 0} = 2$$

Note: We first used the "*A Function of a Sequence***"** theorem from above when we pulled the limit into the square root. We also used the "*Sequence Defined by a Function*" from above by replacing n with x, which allowed us to use the basic limit techniques we have learned in the past. The same techniques will be used in the remainder of the examples, except we will not explicitly show the n being replaced with x.

b.

$$\lim_{n\to\infty}\left(e^{\left(\frac{4n}{3n+9}\right)}\right) = e^{\lim_{n\to\infty}\left(\frac{4n}{3n+9}\right)}$$
$$= e^{\frac{4}{3}\lim_{n\to\infty}\left(\frac{n}{n}\right)}$$
$$= e^{\frac{4}{3}\lim_{n\to\infty}(1)} = e^{\frac{4}{3}}$$

Where, we used techniques we learned in calculus 1 to evaluate the limits at infinity of rational functions.

c.

$$\lim_{n\to\infty}\left(cos\left(\frac{n^2}{2n^4+3n+4}\right)\right) = cos\left(\lim_{n\to\infty}\left(\frac{n^2}{2n^4+3n+4}\right)\right)$$
$$= cos\left(\frac{1}{2}\lim_{n\to\infty}\left(\frac{n^2}{n^4}\right)\right)$$
$$= cos\left(\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right)$$
$$= cos\left(\frac{1}{2}\cdot 0\right)$$
$$= cos(0) = 1$$

d.

$$\lim_{n\to\infty}(tan^{-1}(e^{-n}+1)) = tan^{-1}\left(\lim_{n\to\infty}(e^{-n}+1)\right)$$
$$= tan^{-1}\left(\lim_{n\to\infty}(e^{-n}) + \lim_{n\to\infty}(1)\right)$$
$$= tan^{-1}(0+1)$$
$$= tan^{-1}(1) = \frac{\pi}{4}$$

Example 8: Determine the limit of the given sequences or state that the sequence diverges.

a. $a_n = \frac{n}{ln(n)}$ *b.* $a_n = \frac{n^2}{e^{2n}}$

Solution:

a. In this case we can use L 'Hopital's Rule to evaluate.

$$\lim_{n\to\infty}\left(\frac{n}{ln(n)}\right) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}(n)}{\frac{d}{dn}(ln(n))}\right) = \lim_{n\to\infty}\left(\frac{1}{1/n}\right) = \lim_{n\to\infty}(n) = \infty$$

Therefore, the sequence diverges to infinity.

b.

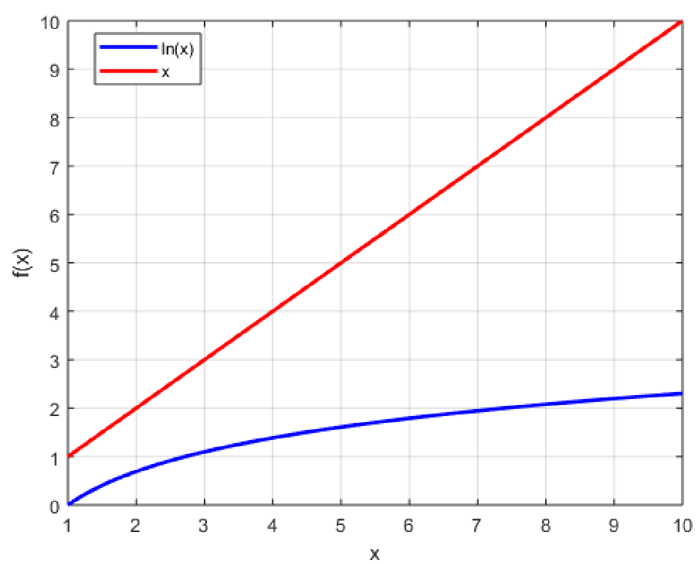
Applying L 'Hopital's Rule again, twice this time, we have

$$\lim_{n\to\infty}\left(\frac{n^2}{e^{2n}}\right) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}(n^2)}{\frac{d}{dn}(e^{2n})}\right) = \lim_{n\to\infty}\left(\frac{2n}{2e^{2n}}\right) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}(2n)}{\frac{d}{dn}(2e^{2n})}\right) = \lim_{n\to\infty}\left(\frac{2}{4e^{2n}}\right) = 0$$

In example 8 we used L 'Hopital's Rule to evaluate the convergence of sequences. By taking the derivative of the numerator and denominator separately this rule in essence is comparing the rate of change of these two terms. For example, if the rate of growth of the numerator increases at a higher rate than the denominator the function will tend to infinity. On the other hand, if the rate of growth of the denominator increases at a higher rate than the numerator the function will tend to zero. Let's look at the first sequence from example 8 to demonstrate.

$$\lim_{n\to\infty}\left(\frac{n}{ln(n)}\right) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}(n)}{\frac{d}{dn}(ln(n))}\right) = \lim_{n\to\infty}\left(\frac{1}{1/_n}\right)$$

We can say that the growth rate of the numerator is 1, whereas the growth rate of the denominator $1/_n$. Therefore, the numerator will continue to grow at a fixed rate while the denominator term will tend to a fixed value, i.e. stop growing, as n gets very large. Because of this fact we see that the function will tend to infinity. The figure below shows the two functions plotted together for illustration.



With this knowledge we can, in some cases, use the following guidelines to help us evaluate limits of rational functions.

$$\lim_{n\to\infty}\left(\frac{higher\ growth\ rate\ function}{lower\ growth\ rate\ function}\right) = \infty$$

$$\lim_{n\to\infty}\left(\frac{lower\ growth\ rate\ function}{higher\ growth\ rate\ function}\right) = 0$$

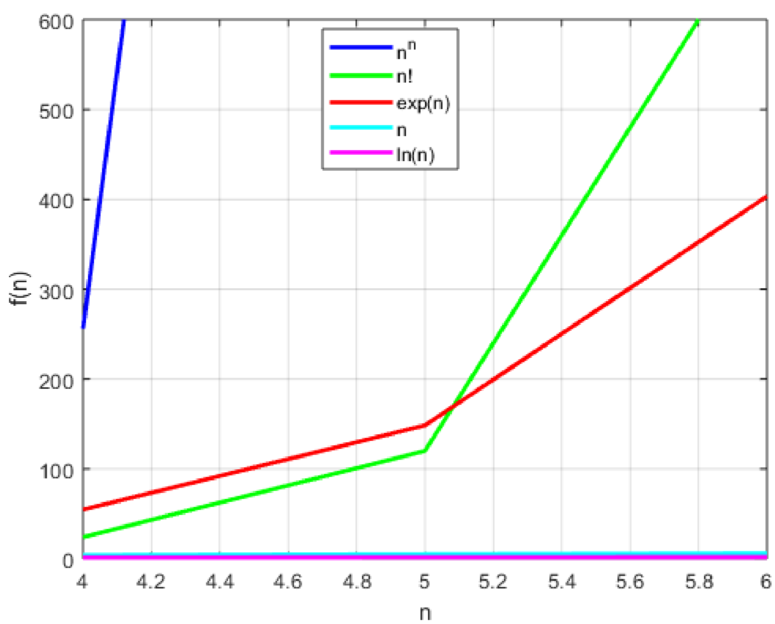
We can also classify the relative growth rate of some common functions as shown below. As n gets very large the following is true.

$$ln(n) \ll n^a \ll b^n \ll n! \ll n^n$$

For $a > 0$ and $b > 1$.

Recall the second sequence from example 8 which contained an exponential function in the denominator and a power function in the numerator. Based on the above discussion an exponential function has a higher growth rate than the power function and therefore we would say that this sequence will tend to zero. As you can see this is the same result we obtained by applying L 'Hopital's Rule.

Finally, for illustration purposes, the figure below shows a small window of the above five functions with $a = 1$ and $b = e$.



Final Summary for Infinite Series – Sequences

Sequence
A sequence, $\{a_n\}$, is an ordered collection of numbers that may or may not be defined by a function, f, on a set of sequential integers. The values a_n are called the terms of the sequence, and n is called the index. $$\{a_n\} = a_1, a_2, a_3, \ldots$$ If the sequence is defined by a function, we can say that $$a_n = f(n)$$ Note: The sequence does not have to start at $n = 1$. It can start at any other integer.
Limit of a Sequence
We say that the sequence, $\{a_n\}$, converges to a limit L, and we write $$\lim_{n\to\infty} a_n = L$$ if, for every $\varepsilon > 0$, there is a number M such that $\lvert a_n - L\rvert < \epsilon$ for all $n > M$. • If no limit exists, we say that $\{a_n\}$ diverges. If the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.
Sequence Defined by a Function
If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit: $$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$
A Function of a Sequence
If f is continuous and $\lim_{n\to\infty}(a_n) = L$, then $$\lim_{n\to\infty}(a_n) = f\left(\lim_{n\to\infty}(a_n)\right) = f(L)$$
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Monotonic Sequences
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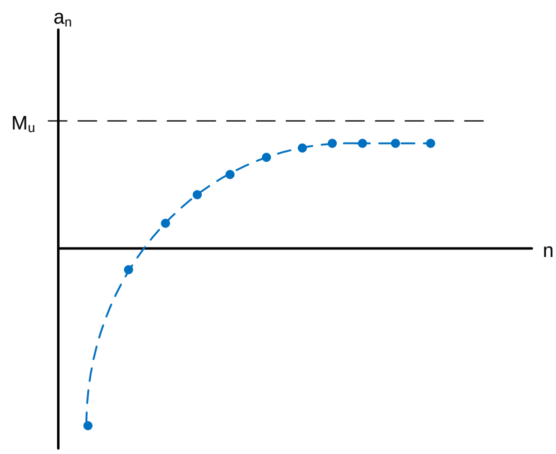

A sequence $\{a_n\}$ is:

- **Monotonically decreasing,** $a_{n+1} < a_n$, and
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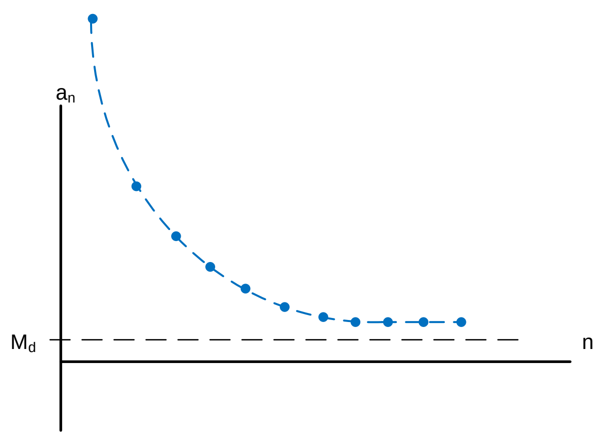
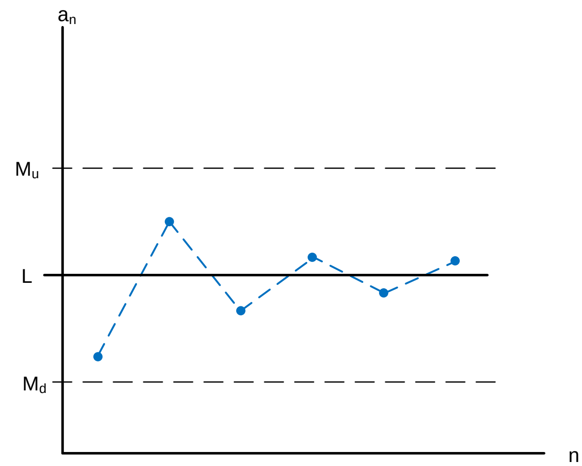
Then $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n) \geq M_d$



Convergent Sequences are Bounded

If $\{a_n\}$ converges, then it is bounded from above and below.



Note: This theorem does **not** state that all bounded sequences converge.

Function Growth Rates

In some cases, we can use the following guidelines to help us evaluate limits of rational functions.

$$\lim_{n\to\infty}\left(\frac{higher\ growth\ rate\ function}{lower\ growth\ rate\ function}\right) = \infty$$

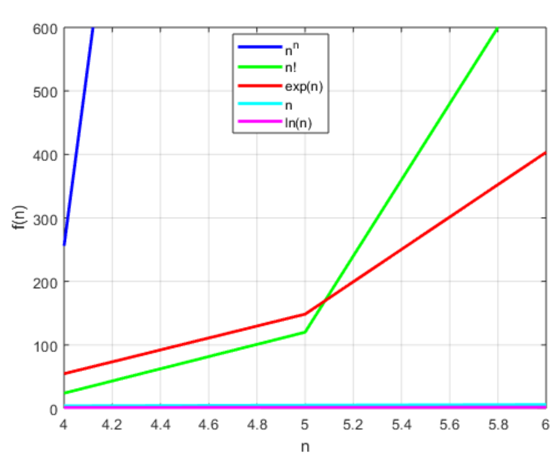
$$\lim_{n\to\infty}\left(\frac{lower\ growth\ rate\ function}{higher\ growth\ rate\ function}\right) = 0$$

We can also classify the relative growth rate of some common functions as shown below. As n gets very large the following is true.

$$ln(n) \ll n^a \ll b^n \ll n! \ll n^n$$

For $a > 0$ and $b > 1$.

Illustrative Plot



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