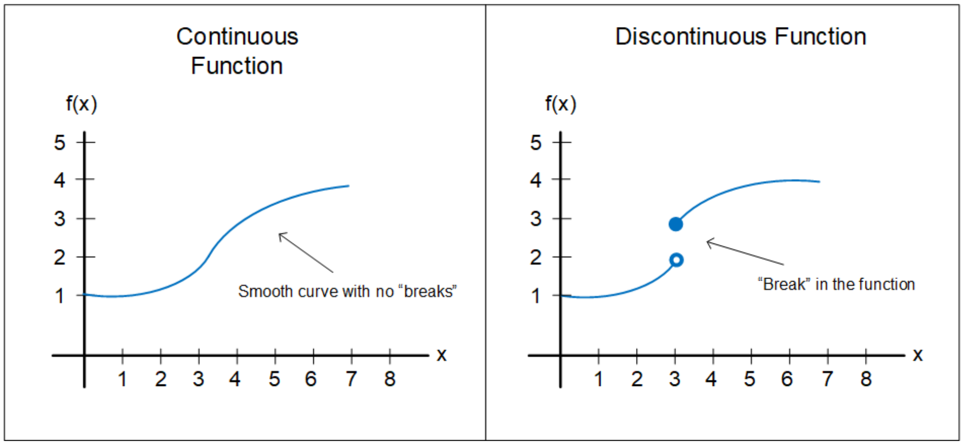
Limits – Continuity

The first major topic in calculus is finding the rate of change of functions, which we briefly introduced in the motivation section. As we shall see later, computing the rate of change of a function at any point requires the function be continuous at that point. In plain terms a continuous function is one that has no breaks in it. The concept is illustrated in the figure below. Of course, continuity can be defined more rigorously and that is what we set out to do in this section.



A formal definition for continuity of a function at a point, c, is given below.

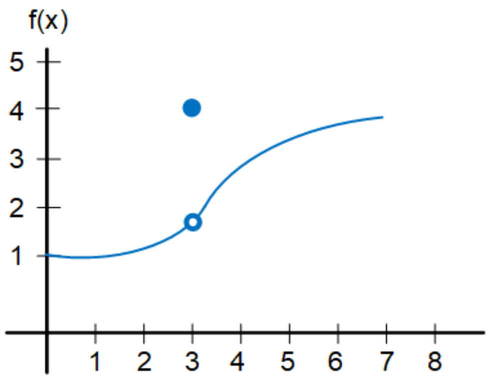
Continuity at a Point
A function, $f(x)$, is continuous at a point, c, if $$\lim_{x \to c}\{f(x)\} = f(c)$$ Otherwise, the function is discontinuous at $x = c$.

The definition above can be distilled into the following three conditions that must be satisfied for a function to be continuous at the point, c.

1. $f(c)$ is defined
2. $\lim_{x \to c}\{f(x)\}$ exists
3. The values of 1 and 2 are equal.

There are three main types of discontinuities which we review below.

1. ***Removable Discontinuity:*** Occurs if $\lim_{x \to c}\{f(x)\}$ exists but is not equal to $f(c)$. An example is shown below.

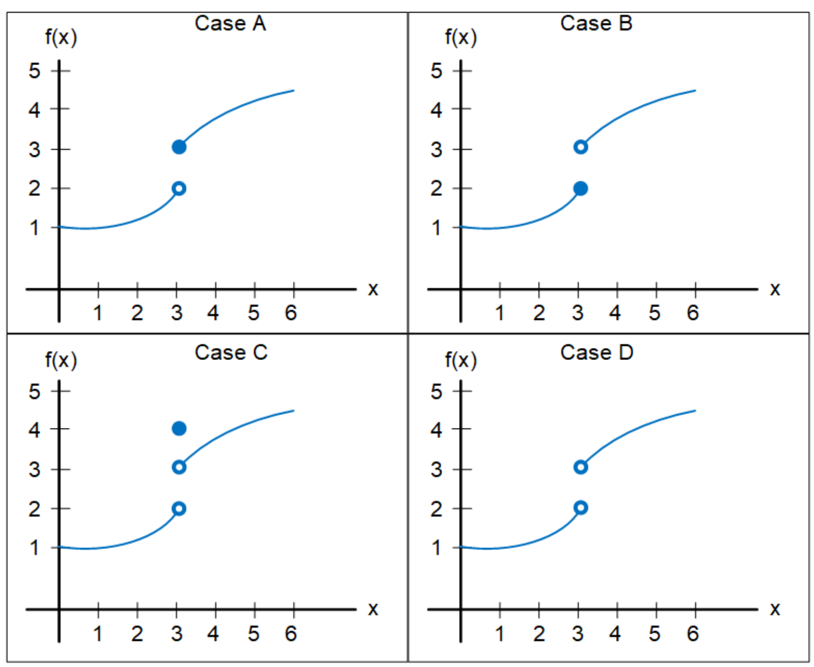


There is a removable discontinuity at $x = 3$ since $f(3) \neq \lim_{x \to 3}\{f(x)\}$

$$f(3) = 4$$

$$\lim_{x \to 3}\{f(x)\} = 2$$

2. ***Jump Discontinuity:*** Occurs when $\lim_{x \to c^-}\{f(x)\}$ and $\lim_{x \to c^+}\{f(x)\}$ both exists but are not equal, and therefore $\lim_{x \to c}\{f(x)\}$ does not exists. Note this definition says nothing about $f(c)$, therefore we can have various cases as shown below.



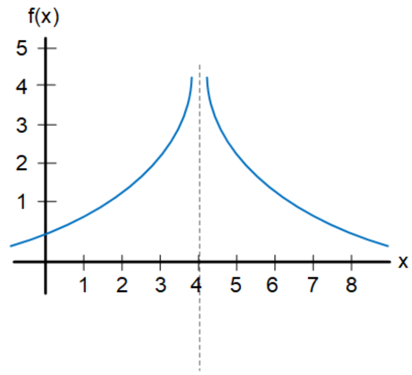
Case A: $\lim_{x \to 3^-}\{f(x)\} = 2,\ \lim_{x \to 3^+}\{f(x)\} = 3,\ f(3) = 3$

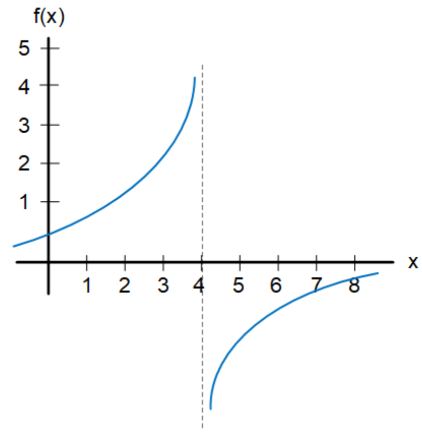
Case B: $\lim_{x \to 3^-}\{f(x)\} = 2,\ \lim_{x \to 3^+}\{f(x)\} = 3,\ f(3) = 2$

Case C: $\lim_{x \to 3^-}\{f(x)\} = 2,\ \lim_{x \to 3^+}\{f(x)\} = 3,\ f(3) = 4$

Case D: $\lim_{x \to 3^-}\{f(x)\} = 2,\ \lim_{x \to 3^+}\{f(x)\} = 3,\ f(3) = DNE$

3. ***Infinite Discontinuity:*** Occurs when $\lim_{x\to c^-}\{f(x)\}$, $\lim_{x\to c^+}\{f(x)\}$, or both are infinite. Two examples are shown below.





Having introduced examples of discontinuities we can now ask the question: "How can we recognize when a function is continuous?" One way to do this is to split a function into various elementary functions that we already know are continuous and then use what we call the laws of continuity. Below is a list of some elementary functions that we know are continuous.

- Polynomials
- $\sin(x)$ and $\cos(x)$
- Exponential functions
- Logarithmic functions on its domain
- $x^{\frac{1}{n}}$ on its domain where n is any natural number.

The laws of continuity, as stated below, can then be used to help determine whether more complex functions are continuous.

Laws of Continuity	
If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:	
1. $f(x) \mp g(x)$	2. $f(x) \cdot g(x)$
3. $kf(x)$, for any k	4. $f(x)/g(x)$, if $g(c) \neq 0$

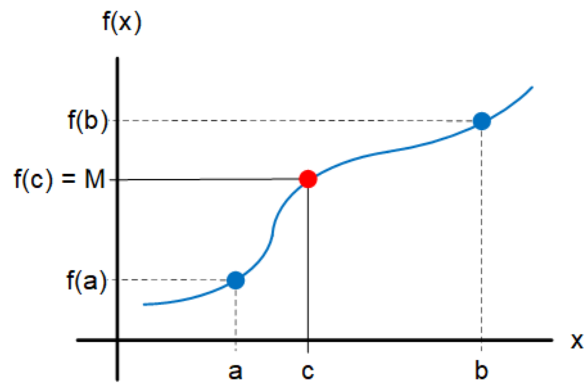
Composite functions are also common ways to build complex functions from elementary ones. Fortunately, the following theorem can also be used.

Continuity of Composite Functions
If $g(x)$ is continuous at $x = c$, and $f(x)$ is continuous at $x = g(c)$, then $F(x)$ is continuous at $x = c$, where: $$F(x) = f\big(g(x)\big)$$

Intermediate Value Theorem

Before moving on to some examples we introduce one last theorem that applies to continuous functions. It's called the intermediate value theorem and in plain terms it says that a continuous function does not skip values as it varies between two values. The formal definition is given below with an accompanying figure for illustration purposes.

Intermediate Value Theorem
If $f(x)$ is continuous on a closed interval, $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value, $c \in (a, b)$ such that $f(c) = M$.



Examples:

Question 1:

Determine the points of discontinuity and state the type of discontinuity for the two functions below.

a.) $f(x) = \frac{x+1}{4x-2}$

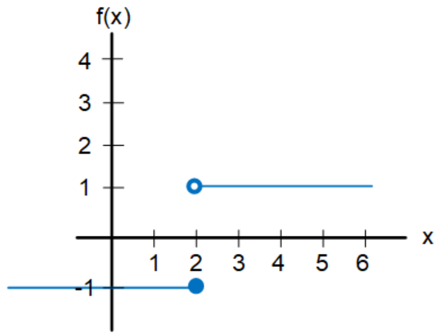
b.) $f(x) = \begin{cases} \frac{x-2}{|x-2|}, & x \neq 2 \\ -1, & x = 2 \end{cases}$

a.) In this case we have a rational function, which consists of a polynomial in the numerator and denominator. Since we know polynomials are continuous everywhere, we can use the following statement from the laws of continuity to find any points of discontinuity.

$$f(x)/\mathrm{g}(x), \text{ if } \mathrm{g}(c) \neq 0$$

Therefore, the function is discontinuous when $\mathrm{g}(x) = 4x - 2 = 0$, or $x = \frac{1}{2}$. From our knowledge of rational functions, we also know that this point corresponds to a vertical asymptote, and therefore is an infinite discontinuity.

b.) For the piecewise function given it is most convenient to plot the function to determine any discontinuities.



The figure shows a jump discontinuity at $x = 2$ since $\lim_{x \to c^-}\{f(x)\} \neq \lim_{x \to c^+}\{f(x)\}$. Furthermore since $\lim_{x \to c^-}\{f(x)\} = f(2) = -1$ we can state that the function is left-continuous.

Question 2:

Find c in the piecewise function below so that the function is continuous.

$$f(x) = \begin{cases} x^2 - c, & x < 5 \\ 4x + 2c, & x \geq 5 \end{cases}$$

The function switches definitions at $x = 5$, and in order for the function to be continuous we need both functions to evaluate to the same value at $x = 5$.

$$5^2 - c = 4 \cdot 5 + 2c$$
$$25 - 20 = 3c$$
$$\frac{5}{3} = c$$

Question 3:

Use the IVT to prove that the following function has a root in the interval $[0,2]$.

$$f(x) = x^3 - \frac{x^2 + 2}{\cos(x) + 2}$$

Using the laws of continuity, we can first show that the function is continuous, at least in the interval $[0,2]$. The first term, x^3, as well as the numerator in the second term are polynomial functions and therefore are continuous. The denominator in the second term consists of a cosine function and a constant term, both of which are continuous, and using the summation law of continuity the denominator is also continuous. Finally, we need to check if there are any x values such that the denominator in the second term is zero.

$$\cos(x) + 2 = 0$$
$$\cos(x) = -2$$

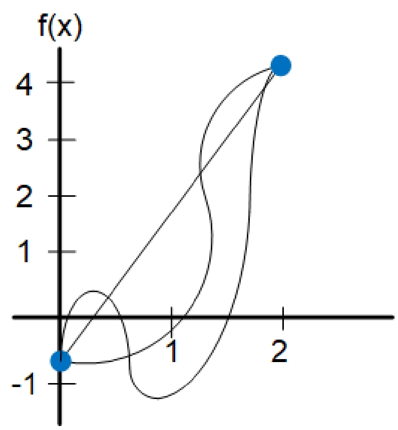
Which has no solutions and therefore the function $f(x)$ is continuous everywhere.

Since the function is continuous on $[0,2]$ the next step is to evaluate the function at $x = 0$ and $x = 2$.

$f(0) = 0^3 - \frac{0^2+2}{\cos(0)+2}$ $f(0) = -\frac{2}{3}$	$f(2) = 2^3 - \frac{2^2+2}{\cos(2)+2}$ $f(2) = 8 - \frac{6}{-0.416+2}$ $f(2) = 4.21$

Finally, from IVT we can state the following:

Since $f(x)$ is continuous on $[0,2]$, $f(0)$ is negative, and $f(2)$ is positive, there must be an x value between 0 and 2 where the function evaluates to zero, (and to all other values between $f(0)$ and $f(2)$). Although we can directly plot this function over $[0,2]$, we show three possible ways the function can vary between $f(0)$ and $f(2)$ to better illustrate the fact that any of these continuous functions will cross the x axis at least once.



Final Summary for Limits – Continuity

Continuity at a Point

A function, $f(x)$, is continuous at a point, c, if

$$\lim_{x \to c}\{f(x)\} = f(c)$$

Otherwise, the function is discontinuous at $x = c$.

Conditions for Continuity of a Function

1. $f(c)$ is defined
2. $\lim_{x \to c}\{f(x)\}$ exists
3. The values of 1 and 2 are equal.

Types of Discontinuities

1. ***Removable Discontinuity:*** Occurs if $\lim_{x \to c}\{f(x)\}$ exists but is not equal to $f(c)$.

2. ***Jump Discontinuity:*** Occurs when $\lim_{x \to c^-}\{f(x)\}$ and $\lim_{x \to c^+}\{f(x)\}$ both exists but are not equal, and therefore $\lim_{x \to c}\{f(x)\}$ does not exist.

3. ***Infinite Discontinuity:*** Occurs when $\lim_{x \to c^-}\{f(x)\}$, $\lim_{x \to c^+}\{f(x)\}$, or both are infinite.

Common Continuous Functions

- Polynomials
- $\sin(x)$ and $\cos(x)$
- Exponential functions
- Logarithmic functions on its domain
- $x^{\frac{1}{n}}$ on its domain where n is any natural number.

Laws of Continuity

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Intermediate Value Theorem

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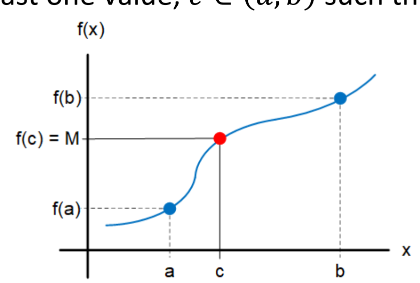

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