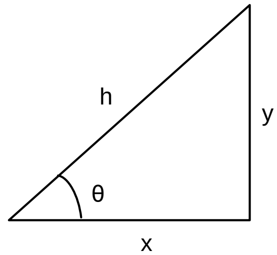
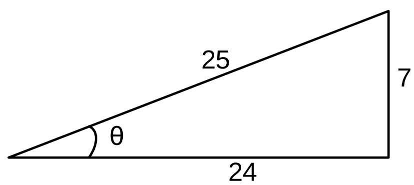
Pre-Calculus Trigonometry Review

Our trigonometry review starts with a review of *right triangle trigonometry*. With respect to a right triangle we define six trigonometric functions that relate one of the angles (other than the 90° angle) to the various ratios of the side lengths. See the six definitions below as they relate to the triangle shown.



$\sin(\theta) = \dfrac{opposite}{hypotenuse} = \dfrac{y}{h}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{hypotenuse}{opposite} = \dfrac{h}{y}$
$\cos(\theta) = \dfrac{adjacent}{hypotenuse} = \dfrac{x}{h}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{hypotenuse}{adjacent} = \dfrac{h}{x}$
$\tan(\theta) = \dfrac{opposite}{adjacent} = \dfrac{y}{x}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{adjacent}{opposite} = \dfrac{x}{y}$

Let's look at the following example:

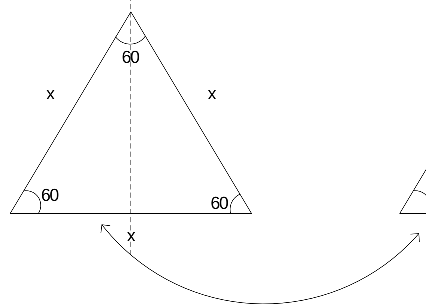


Using the side lengths given, the six trigonometric functions are evaluated as follows:

$\sin(\theta) = \dfrac{opposite}{hypotenuse} = \dfrac{7}{25}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{25}{7}$
$\cos(\theta) = \dfrac{adjacent}{hypotenuse} = \dfrac{24}{25}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{25}{24}$
$\tan(\theta) = \dfrac{opposite}{adjacent} = \dfrac{7}{24}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{24}{7}$

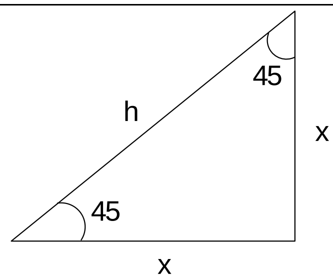
Special Angles:

Using an equilateral triangle and a $45^\circ/45^\circ$ right triangle we can derive trigonometric values for three fundamental angles, $30^\circ, 45^\circ$, and 60°. We start by cutting an equilateral triangle in half, as shown below, and find the relationship between y and x using the Pythagorean theorem.



$$y = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}$$
$$y = x\sqrt{1 - \frac{1}{4}}$$
$$y = x\frac{\sqrt{3}}{2}$$

Next, we look at a $45^{\circ} - 45^{\circ}$ right triangle. Since these triangles have equal side lengths, we can easily find the hypotenuse, again using Pythagorean theorem.

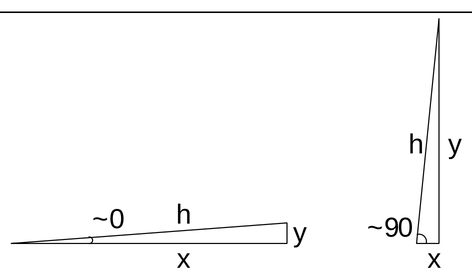


$$h = \sqrt{x^2 + x^2}$$
$$h = x\sqrt{2}$$

Note that in both cases above all sides are defined in terms of x. Therefore, for any ratio of side lengths the x term will cancel, and we can easily evaluate all six trigonometric functions. Examples for the sine function are shown below.

$\sin(30^{\circ}) = \frac{\frac{\cancel{x}}{2}}{\cancel{x}} = \frac{1}{2}$	$\sin(60^{\circ}) = \frac{\cancel{x}\frac{\sqrt{3}}{2}}{\cancel{x}} = \frac{\sqrt{3}}{2}$	$\sin(45^{\circ}) = \frac{\cancel{x}}{\cancel{x}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Lastly, we note two other "*special*" angles, 0° and 90°. The figures below show what these triangles look like when the angles approach 0° and 90°. Without being rigorous we could easily imagine that when $\theta = 0^{\circ}$ we will have $x = h$ and $y = 0$, and when $\theta = 90^{\circ}$ we will have $y = h$, and $x = 0$.



When $\theta = 0^{\circ}$, $x = h$, and $y = 0$

When $\theta = 90^{\circ}$, y= h, and $x = 0$

Finally, the table below lists the values of all six trigonometric functions for the special angles discussed above.

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$	$\mathbf{tan}(\boldsymbol{\theta})$	$\mathbf{csc}(\boldsymbol{\theta})$	$\mathbf{sec}(\boldsymbol{\theta})$	$\mathbf{cot}(\boldsymbol{\theta})$
$\mathbf{0^\circ}$	0	1	0	∞	1	∞
$\mathbf{30^\circ}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\mathbf{45^\circ}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\mathbf{60^\circ}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{2}$
$\mathbf{90^\circ}$	1	0	∞	1	∞	0

Committing this table to memory will prove helpful and luckily is not as difficult as it may first seem. First notice that the sine and cosine functions are the most important trigonometric functions in the sense that all others function values can be derived from their values. In other words, once we know that values of $\sin(\theta)$ and $\cos(\theta)$ the others can be determined as follows:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Therefore, with these two columns known we can reproduce the entire table. Secondly notice that the cosine values are just a reverse reading of the sine values. Therefore, we need only really memorize one column from this entire table! Finally, if we rewrite the values for $\sin(\theta)$ in a different format we notice a very simple pattern. Examining the table below we see that each entry has a factor of $1/2$ multiplied by the square root of the numbers 0 through 4. Verify for yourself that the values shown below match the table from above.

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$
$\mathbf{0^\circ}$	$\left(\frac{1}{2}\right)\sqrt{0}$	$\left(\frac{1}{2}\right)\sqrt{4}$
$\mathbf{30^\circ}$	$\left(\frac{1}{2}\right)\sqrt{1}$	$\left(\frac{1}{2}\right)\sqrt{3}$
$\mathbf{45^\circ}$	$\left(\frac{1}{2}\right)\sqrt{2}$	$\left(\frac{1}{2}\right)\sqrt{2}$
$\mathbf{60^\circ}$	$\left(\frac{1}{2}\right)\sqrt{3}$	$\left(\frac{1}{2}\right)\sqrt{1}$
$\mathbf{90^\circ}$	$\left(\frac{1}{2}\right)\sqrt{4}$	$\left(\frac{1}{2}\right)\sqrt{0}$

Angles greater than 90°

To extend the trigonometric functions beyond 90°we start by placing our right triangle in an $x - y$ coordinate plane and reflecting it across the x and y axis so that we have a triangle in each quadrant. Note that scaling a triangle, i.e. multiplying all sides lengths by a constant, does not change the interior angles and hence does not change the value of any of the trigonometric functions. Therefore, we can set the hypotenuse to one so that the sine and cosine functions correspond directly to the length of the triangle in the y and x direction respectively.

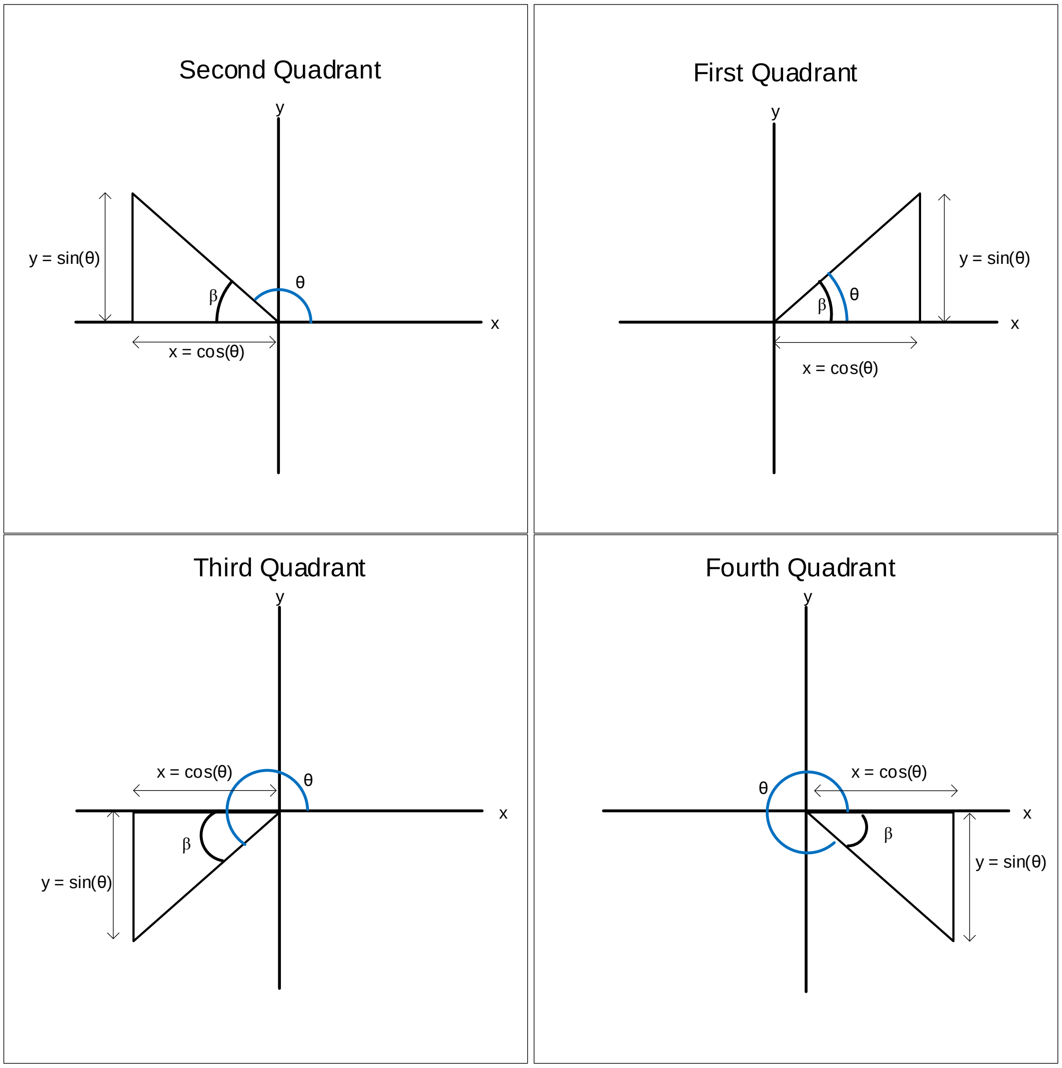
$$\sin(\theta) = y, \quad \cos(\theta) = x$$

The special angles we developed above, which correspond to a triangle in the first quadrant, can now be extended to the other three quadrants. The convention is that the angle is measured in the counterclockwise direction from the positive x axis to the hypotenuse of the triangle. To find values of sine and cosine for all quadrants, i.e. θ varies from 0° to 360°, we proceed as follows:

1. Find the angle β, which we call the reference angle.
2. Evaluate $\sin(\beta)$ and $\cos(\beta)$ to find the magnitude of the value.
3. Determine the sign of the value as required based on which quadrant the angle, θ, exists.

The procedure for finding the reference angle and determining the sign of the value for each quadrant is explained below. You should verify for yourself by looking at the figures as you read the explanations below.

- First Quadrant
    - Reference angle, $\beta = \theta$
    - All sine and cosine values are positive.
- Second Quadrant
    - $\beta = 180^{\circ} - \theta$
    - Cosine values are negative, while sine values remain positive.
- Third Quadrant
    - $\beta = \theta - 180^{\circ}$
    - All sine and cosine values are negative.
- Fourth Quadrant
    - $\beta = 360^{\circ} - \theta$
    - Sine values are negative, while cosine values remain positive.


Second Quadrant
y
θ
β
y = sin(θ)
x = cos(θ)
x
First Quadrant
y
θ
β
y = sin(θ)
x = cos(θ)
x
Third Quadrant
y
x = cos(θ)
θ
β
y = sin(θ)
x
Fourth Quadrant
y
θ
x = cos(θ)
β
y = sin(θ)
x

Finally, the table below shows the values for the special angles of the sine, cosine and tangent functions as θ varies from 0° to 360°

Quadrant	$\boldsymbol{\theta}$	$\boldsymbol{\beta}$	$\mathbf{sin(\boldsymbol{\theta})}$	$\mathbf{cos(\boldsymbol{\theta})}$	$\mathbf{tan(\boldsymbol{\theta})}$
First Quadrant (x and y positive)	0°	0°	0	1	0
	30°	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	45°	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
	60°	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
	90°	90°	1	0	∞
Second Quadrant (x negative, y positive)	120°	60°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
	135°	45°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
	150°	30°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	180°	0°	0	-1	
Third Quadrant (x and y negative)	210°	30°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	225°	45°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
	240°	60°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
	270°	90°	-1	0	$-\infty$
Fourth Quadrant (x positive, y negative)	300°	60°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
	315°	45°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
	330°	30°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	360°	0°	0	1	0

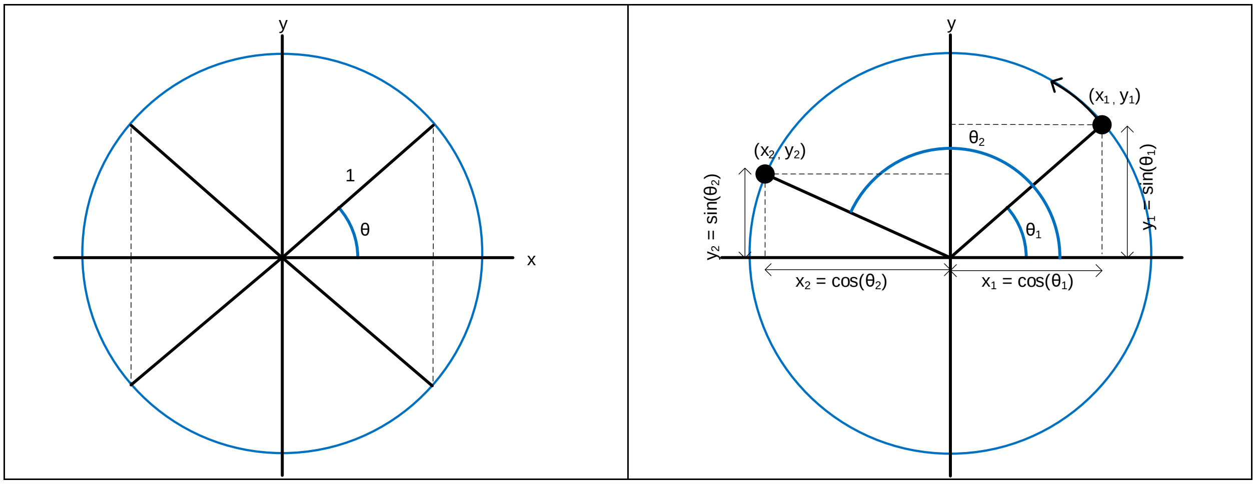
So far, we have shown all angles being measured in degrees. As you may recall it is also common to measure angles in *radians*, where $360^\circ = 2\pi$ radians.

The Unit Circle and Graphs of the Trigonometric Functions:

In the previous section we defined the trigonometric functions for $0^\circ \leq \theta \leq 360^\circ$ by drawing a triangle in each quadrant of a 2D coordinate plane. We can extend these functions even further while at the same time removing the anchor of the right triangle all together. We illustrate this process with the figures below by placing all four triangles on the same 2D plane showing only the hypotenuse of each of the triangles. When we do this, we notice that the hypotenuse is just a line that measures the radius of a circle, (a unit circle when the hypotenuse is one). Next, we can imagine this line "points" to a particle that is rotating around this circle in the counterclockwise direction. At any point in time we can define the location of this particle with an $x - y$ coordinate point, e.g. $\big(x(t), y(t)\big)$. Finally, we notice that for a unit circle these coordinates correspond directly to the sine and cosine functions, e.g. $\big(\cos\big(\theta(t)\big), \sin\big(\theta(t)\big)\big)$! Therefore, we can now define the sine and cosine functions as the x and y coordinates of a particle that is traveling around a circle of radius A as shown.

$$x(t) = A\cos\big(\theta(t)\big) \qquad\qquad y(t) = A\sin\big(\theta(t)\big)$$

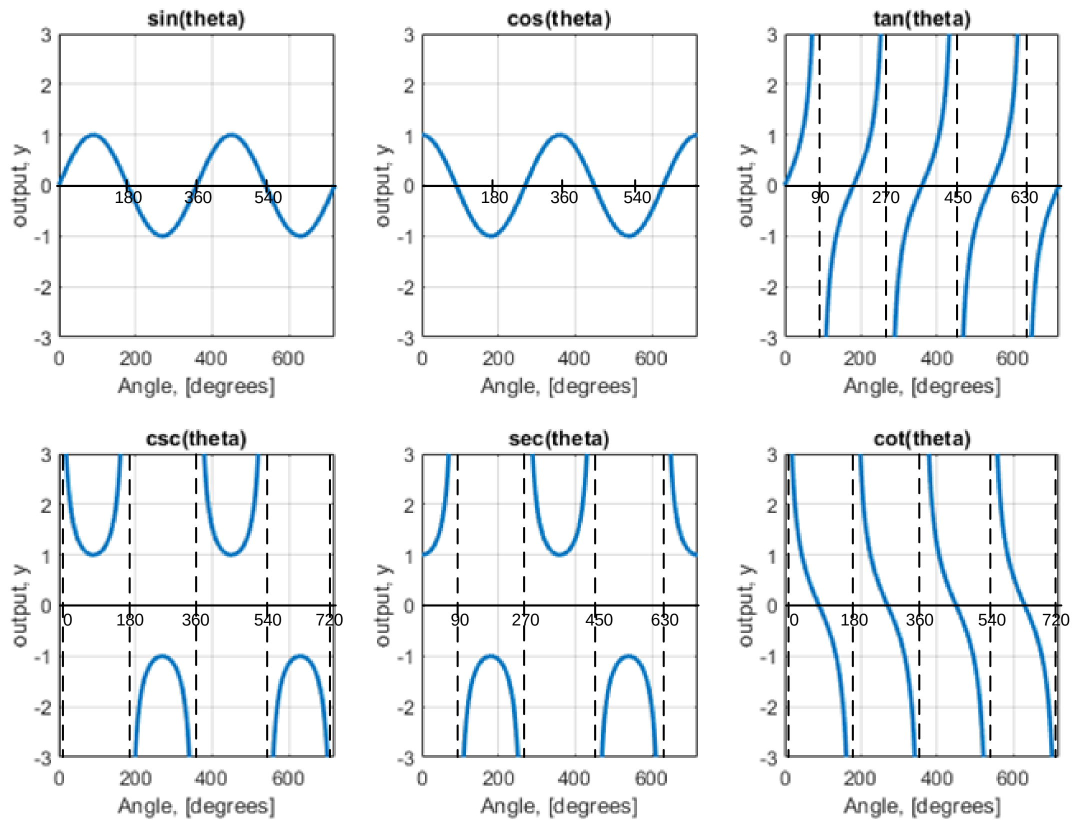
Where, A is the radius of the circle.



This new interpretation allows us to define the trigonometric functions for all values of θ from $-\infty$ to ∞. Note these functions are periodic since they will repeat every time the particle makes a complete revolution around the circle, which we call the period, T, measured in seconds per cycle. Furthermore, we define the frequency as $f = 1/T$, measured in cycles per seconds. If we assume a particle starts at an angle of θ_0 and travels around a circle of radius A with a frequency of f Hertz, the position of this particle at any time, t, is given as shown below. The equations are shown using degrees and radians. Even though we have been focusing on degrees, its more common to represent these types of equations using radians.

Measuring the angle in degrees	**Measuring the angle in radians**
$x(t) = A\cos(360^\circ ft + \theta_0)$	$x(t) = A\cos(2\pi ft + \theta_0)$
$y(t) = A\sin(360^\circ ft + \theta_0)$	$y(t) = A\sin(2\pi ft + \theta_0)$

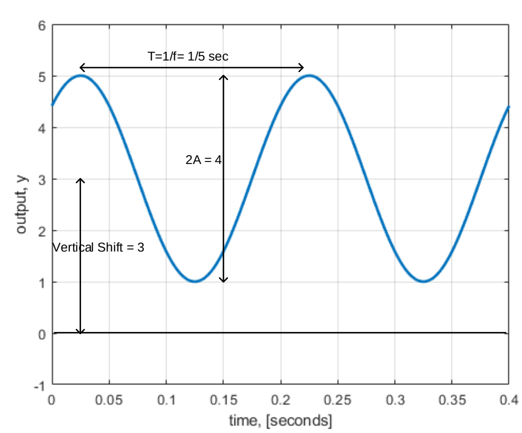
With the above interpretation we can abandon the triangle idea and treat trigonometric functions as other functions that we can plot in a 2D plane. Graphs for the six trigonometric functions are shown below.



The above figures show all functions plotted using the angle, θ, as the independent variable. As we mentioned above, we can also express this angle as a function of time. e.g. $\theta(t) = 2\pi f t + \theta_0$, where $\theta(t)$ is in radians. As example we can have.

$$y(t) = 2\sin(2\pi(5)t + \pi/4) + 3$$

The graph below shows the above sine wave with an amplitude of 2, a frequency of 5, vertical shift of 3, and a left phase shift of $\pi/4$ radians, (or equivalently a time shift of $1/40$ seconds).



Trigonometric Identities:

As mentioned above the trigonometric functions are periodic functions, i.e. the outputs from these functions repeat at fixed intervals. This property is most obvious by returning to the particle traveling along a circle shown above. Because these functions are both periodic and so interconnected there exists various relationships between them that we call *identities*. We introduce some of the most useful trigonometric identities below.

We start with the most obvious identities that arise directly from the definitions.

Reciprocal Identities		
$\sin(\theta) = \dfrac{1}{\csc(\theta)}$	$\cos(\theta) = \dfrac{1}{\sec(\theta)}$	$\tan(\theta) = \dfrac{1}{\cot(\theta)}$
$\csc(\theta) = \dfrac{1}{\sin(\theta)}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$
$\sin(\theta)\csc(\theta) = 1$	$\cos(\theta)\sec(\theta) = 1$	$\tan(\theta)\cot(\theta) = 1$

Quotient Identities	
$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

The next set of identities can be derived be returning the right triangle interpretation and letting the hypotenuse equal one.

From the Pythagorean Theorem we have:

$$y^2 + x^2 = 1^2$$

And since we also know that $x = \cos(\theta)$ and $y = \sin(\theta)$, we can write the following.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Finally, dividing through by either $\sin^2(\theta)$ or $\cos^2(\theta)$ we can derive two additional identities. Together these are known as the Pythagorean Identities.

Pythagorean Identities		
$\sin^2(\theta) + \cos^2(\theta) = 1$	$1 + \cot^2(\theta) = \csc^2(\theta)$	$\tan^2(\theta) + 1 = \sec^2(\theta)$

The next set of identities arise from the definitions of even and odd functions. Recall a function is even if it is symmetric with respect the y axis, i.e. $f(-x) = f(x)$, and odd if it is symmetric with respect to the origin, $f(-x) = -f(x)$. We can identify each of the six trigonometric functions as either even or odd by looking at the six graphs from above.

Even/Odd Identities		
$\sin(-\theta) = -\sin(\theta)$	$\cos(-\theta) = \cos(\theta)$	$\tan(-\theta) = -\tan(\theta)$
$\csc(-\theta) = -\csc(\theta)$	$\sec(-\theta) = \sec(\theta)$	$\cot(-\theta) = -\cot(\theta)$

The next set of identities, sum and difference identities, can be used to derive all the remaining identities we will discuss. The proof for these can be found in the trigonometry section.

Sum and Difference Identities
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
$\tan(A+B) = \dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$
$\tan(A-B) = \dfrac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

Notice the sum and difference identities from above are used to convert a sinusoid function that has as its argument the addition or subtraction of two angles to an expression involving the product of sinusoids of a single angle only. In some cases, we are presented with a product of sinusoids and would like to convert this to a sum of sinusoids to make subsequent calculations more straightforward. We can use the identities from above to derive new identities, so-called product to sum identities, that can allow us to do just that. We will demonstrate the procedure by subtracting the cosine identities to derive the first product to sum identity. We will leave the remaining derivations as an exercise.

$$\sin(A+B) + \sin(A-B) = \sin(A)\cos(B) + \cancel{\cos(A)\sin(B)} + \sin(A)\cos(B) - \cancel{\cos(A)\sin(B)}$$
$$\sin(A+B) + \sin(A-B) = \sin(A)\cos(B) + \sin(A)\cos(B)$$
$$\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$$
$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Product to Sum Identities
$\sin(A)\sin(B) = \dfrac{1}{2}[\cos(A-B) - \cos(A+B)]$
$\cos(A)\cos(B) = \dfrac{1}{2}[\cos(A-B) + \cos(A+B)]$
$\sin(A)\cos(B) = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$
$\cos(A)\sin(B) = \dfrac{1}{2}[\sin(A+B) - \sin(A-B)]$

The next set of identities are sometimes called power reducing identities and can be found by letting $A = B$ in the first three identities from above.

$$\sin(A)\sin(A) = \frac{1}{2}[\cos(A - A) - \cos(A + A)]$$
$$\sin^2(A) = \frac{1}{2}[1 - \cos(2A)]$$

$$\cos(A)\cos(A) = \frac{1}{2}[\cos(A - A) + \cos(A + A)]$$
$$\cos^2(A) = \frac{1}{2}[1 + \cos(2A)]$$

$$\sin(A)\cos(A) = \frac{1}{2}[\sin(A + A) + \cancel{\sin(A - A)}]$$
$$\sin(A)\cos(A) = \frac{1}{2}[\sin(2A)]$$

We can also derive an identity for the tangent function by dividing the first two equations from above.

$$\frac{\sin^2(A)}{\cos^2(A)} = \frac{\cancel{\frac{1}{2}}[1 - \cos(2A)]}{\cancel{\frac{1}{2}}[1 + \cos(2A)]}$$
$$\tan^2(A) = \frac{[1 - \cos(2A)]}{[1 + \cos(2A)]}$$

The power reducing identities are as follows:

Power Reducing Identities
$\sin^2(A) = \frac{1}{2}[1 - \cos(2A)]$
$\cos^2(A) = \frac{1}{2}[1 + \cos(2A)]$
$\sin(A)\cos(A) = \frac{1}{2}[\sin(2A)]$
$\tan^2(A) = \frac{[1 - \cos(2A)]}{[1 + \cos(2A)]}$

The next set of identities are called double angle identities. The first three are simple rearrangements of the the first three power reducing identities. A fourth one can be derived by subtracting the first two power reducing identities as follows.

$$\frac{1}{2}[1+\cos(2A)] - \frac{1}{2}[1-\cos(2A)] = \cos^2(A) - \sin^2(A)$$
$$\cancel{\frac{1}{2}} + \frac{1}{2}\cos(2A) - \cancel{\frac{1}{2}} + \frac{1}{2}\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

We can also derive a double angle identity for the tangent function by setting $A = B$ in the sum and difference identity as follows.

$$\tan(A+A) = \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)}$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

The double angle identities are as follows:

Double Angle Identities
$\cos(2A) = 1 - 2\sin^2(A)$
$\cos(2A) = 1 + 2\cos^2(A)$
$\cos(2A) = \cos^2(A) - \sin^2(A)$
$\sin(2A) = 2\sin(A)\cos(A)$
$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

The final set of identities are called half-angle identities. The first three are derived by simply letting $A = A/2$ in the first two and last power reducing identities. We can also derive two additional forms of the tangent half-angle identity with help from one of the Pythagorean identities as shown.

$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos(A)]}{[1+\cos(A)]}$$
$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos(A)]}{[1+\cos(A)]} \cdot \frac{[1+\cos(A)]}{[1+\cos(A)]}$$
$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos^2(A)]}{[1+\cos(A)]^2}$$
$$\tan^2\left(\frac{A}{2}\right) = \left[\frac{\sin(A)}{1+\cos(A)}\right]^2$$

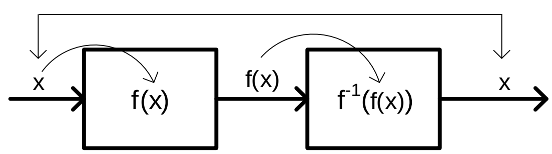
$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos(A)]}{[1+\cos(A)]}$$
$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos(A)]}{[1+\cos(A)]} \cdot \frac{[1-\cos(A)]}{[1-\cos(A)]}$$
$$\tan^2\left(\frac{A}{2}\right) = \frac{[1-\cos^2(A)]^2}{[1-\cos(A)]^2}$$
$$\tan^2\left(\frac{A}{2}\right) = \left[\frac{1-\cos^2(A)}{\sin(A)}\right]^2$$

Finally, the half-angle identities are as follows:

Half Angle Identities
$\sin^2\left(\frac{A}{2}\right) = \frac{1}{2}[1 - \cos(A)]$
$\cos^2\left(\frac{A}{2}\right) = \frac{1}{2}[1 + \cos(2)]$
$\tan^2\left(\frac{A}{2}\right) = \frac{[1 - \cos(A)]}{[1 + \cos(A)]}$
$\tan^2\left(\frac{A}{2}\right) = \left[\frac{\sin(A)}{1 + \cos(A)}\right]^2$
$\tan^2\left(\frac{A}{2}\right) = \left[\frac{1 - \cos^2(A)}{\sin(A)}\right]^2$

Inverse Trigonometric Functions:

Recall that an inverse function is one that reverses the effects of another function. The block diagram below illustrates this concept for a general function.



Let's take a simple linear function, $f(x) = mx + b$, as an example. Recall when finding the inverse, we solve for x in the original function and then re-write this relationship as the inverse function. Using the above equation, we have.

$$x = \frac{f(x) - b}{m}$$

Therefore, the inverse of the function $f(x) = mx + b$ is

$$f^{-1}(x) = \frac{x - b}{m}$$

Now let's see when happens when we apply the output of the original function to the inverse function.

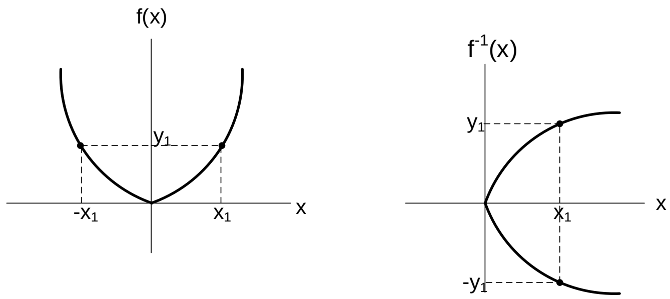
$$f^{-1}\big(f(x)\big) = \frac{f(x) - b}{m}$$
$$f^{-1}\big(f(x)\big) = \frac{mx + \cancel{b} - \cancel{b}}{m}$$
$$f^{-1}\big(f(x)\big) = \frac{\cancel{m}x}{\cancel{m}} = x$$

Therefore, $f^{-1}(x)$ is a valid inverse function.

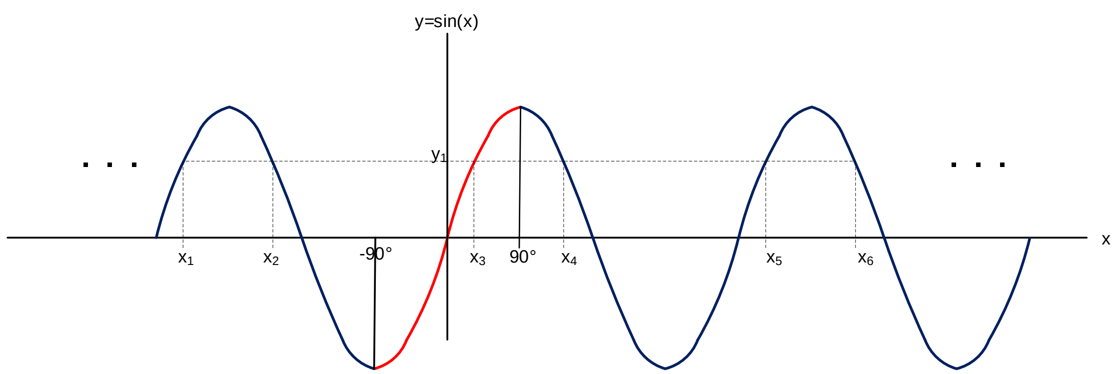
Now let's see what happens if the original function is $f(x) = x^2$. In this case solving for the inverse gives the following.

$$f^{-1}(x) = \mp\sqrt{x}$$

In other words, the inverse is not a valid function in the sense that each input value results in two output values. As you can see from below this result is due to the original function outputting the same value for two separate input values.



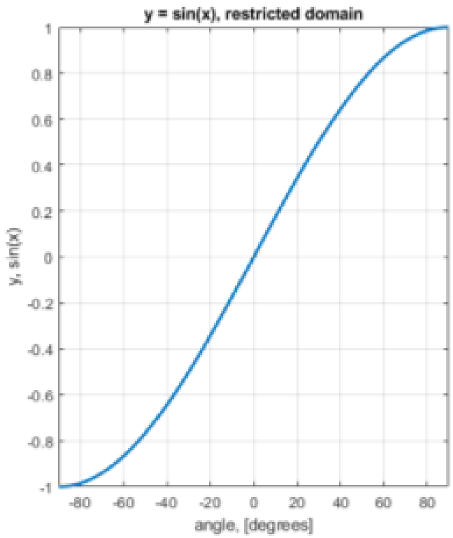
Let's now return to the trigonometric functions. As we already know these functions are periodic and therefore produce an infinite number of identical output values for different inputs! The sine function is shown below for illustrative purposes, where we see the y_1 is repeated for an infinite number of x values.

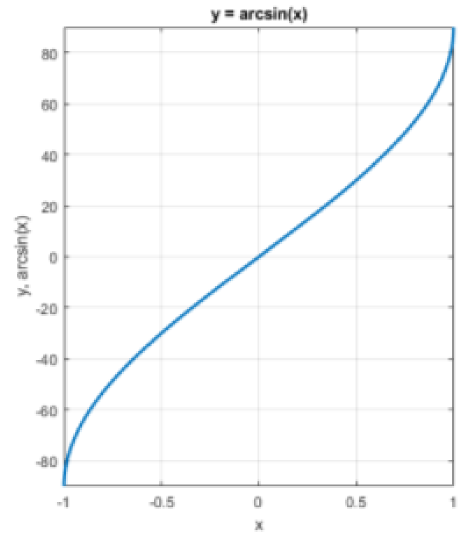


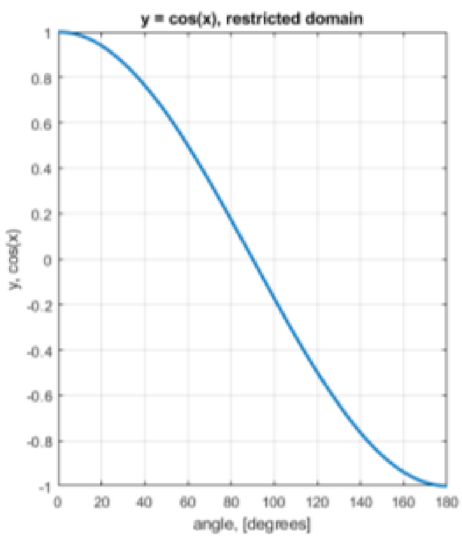
The question is "Can we then define inverse trigonometric functions?" The answer is "Yes", and we do this by restricting the domains of the trigonometric functions such that the functions become one to one. Of course, we can choose any number of domains to satisfy this criterion. The convention that is used is shown below. Note the domain for the sine function is shown as red in the above plot.

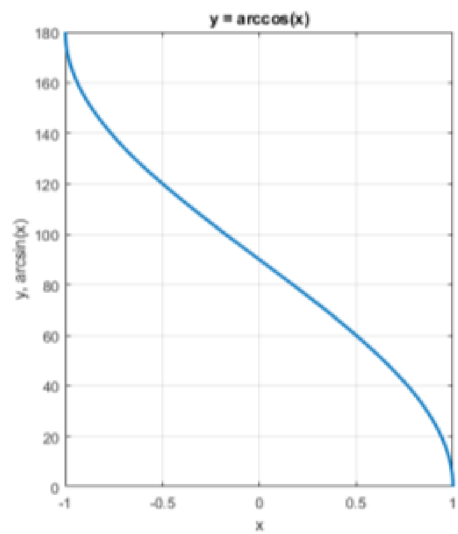
Trigonometric Function	**Restricted Domain**
Sine and Cosecant	$[-90^\circ, 90^\circ]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Cosine and Secant	$[0^\circ, 180^\circ]$ or $[0, \pi]$
Tangent and Cotangent	$[-90^\circ, 90^\circ]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

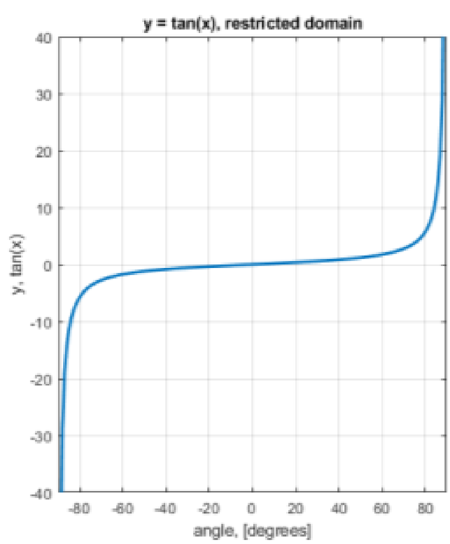
The graphs of the restricted sine, cosine, and tangent functions, along with their inverses, $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$ are shown below.

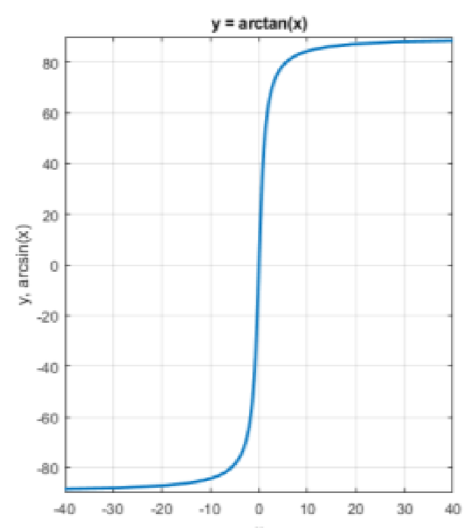






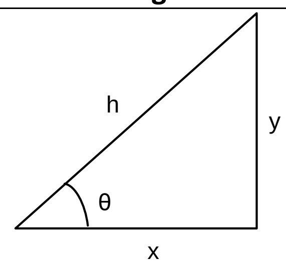






Final Summary for Pre-Calc Trigonometry Review

Right Angle Based Trigonometric Definitions



$\sin(\theta) = \frac{opposite}{hypotenuse} = \frac{y}{h}$	$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{hypotenuse}{opposite} = \frac{h}{y}$
$\cos(\theta) = \frac{adjacent}{hypotenuse} = \frac{x}{h}$	$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{hypotenuse}{adjacent} = \frac{h}{x}$
$\tan(\theta) = \frac{opposite}{adjacent} = \frac{y}{x}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{adjacent}{opposite} = \frac{x}{y}$

Important Sine and Cosine Values in Quadrant I

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$	
$\mathbf{0}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{0}$	$\left(\frac{1}{2}\right)\sqrt{4}$	**Note: The other four trigonometric function values can be derived based on definitions from above.**
$\mathbf{30}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{1}$	$\left(\frac{1}{2}\right)\sqrt{3}$	
$\mathbf{45}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{2}$	$\left(\frac{1}{2}\right)\sqrt{2}$	
$\mathbf{60}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{3}$	$\left(\frac{1}{2}\right)\sqrt{1}$	
$\mathbf{90}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{4}$	$\left(\frac{1}{2}\right)\sqrt{0}$	

Function Values for angles greater that $\mathbf{90}^{\circ}$

Procedure:

1. Find the reference angle, β, based on the quadrant
2. Evaluate the function based on the reference angle to find the magnitude, i.e. $\sin(\beta)$
3. Determine the sign of the values based on the quadrate of the original angle.

Quadrant Rules:

- Quadrant I: $\beta = \theta$
  - Sine and cosine function are both positive.
- Quadrant II: $\beta = 180^{\circ} - \theta$
  - Sine function is positive, cosine function is negative.
- Quadrant III: $\beta = \theta - 180^{\circ}$
  - Sine function is negative, cosine function is negative.
- Quadrant IV: $\beta = 360^{\circ} - \theta$
  - Sine function is negative, cosine function is positve.

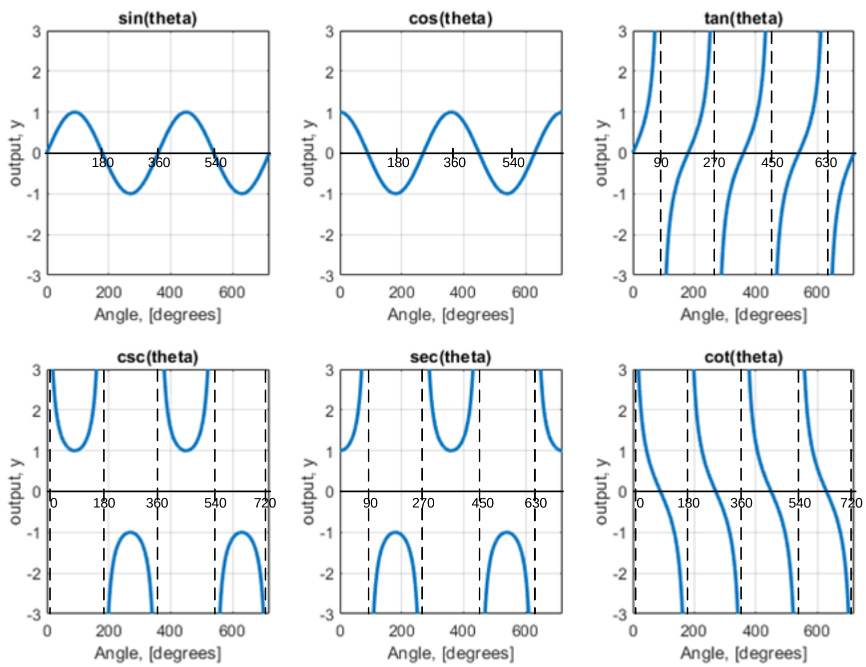
Trigonometric Function Graphs

The x and y coordinates at any time, t, of a particle traveling along a circular path can be described using sine and cosine functions in the most general terms as follows.

$$x(t) = A\cos(2\pi f t + \theta_0) + C$$
$$y(t) = A\sin(2\pi f t + \theta_0) + C$$

Where A is the amplitude, f is the frequency measured in cycles/seconds - or Hertz, $T = \frac{1}{f}$ is the time period measured in seconds/cycle, C is the vertical shift, θ_0 is the phase shift measured in radians, and $\frac{\theta_0}{2\pi f}$ is the time shift measured in seconds.

All other functions can be similarly defined. The graphs for all six trigonometric functions plotted against the angle are shown below.



Trigonometric Identities

Reciprocal Identities		
$\sin(\theta) = \frac{1}{\csc(\theta)}$	$\cos(\theta) = \frac{1}{\sec(\theta)}$	$\tan(\theta) = \frac{1}{\cot(\theta)}$
$\csc(\theta) = \frac{1}{\sin(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$
$\sin(\theta)\csc(\theta) = 1$	$\cos(\theta)\sec(\theta) = 1$	$\tan(\theta)\cot(\theta) = 1$

Quotient Identities	
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities		
$\sin^2(\theta) + \cos^2(\theta) = 1$	$1 + \cot^2(\theta) = \csc^2(\theta)$	$\tan^2(\theta) + 1 = \sec^2(\theta)$

Even/Odd Identities		
$\sin(-\theta) = -\sin(\theta)$	$\cos(-\theta) = \cos(\theta)$	$\tan(-\theta) = -\tan(\theta)$
$\csc(-\theta) = -\csc(\theta)$	$\sec(-\theta) = \sec(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Sum and Difference Identities
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$
$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

Product to Sum Identities
$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
$\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

Trigonometric Identities Cont.

Power Reducing Identities
$\sin^2(A) = \frac{1}{2}[1 - \cos(2A)]$
$\cos^2(A) = \frac{1}{2}[1 + \cos(2A)]$
$\sin(A)\cos(A) = \frac{1}{2}[\sin(2A)]$
$\tan^2(A) = \frac{[1 - \cos(2A)]}{[1 + \cos(2A)]}$

Double Angle Identities
$\cos(2A) = 1 - 2\sin^2(A)$
$\cos(2A) = 1 + 2\cos^2(A)$
$\cos(2A) = \cos^2(A) - \sin^2(A)$
$\sin(2A) = 2\sin(A)\cos(A)$
$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$

Half Angle Identities
$\sin^2\left(\frac{A}{2}\right) = \frac{1}{2}[1 - \cos(A)]$
$\cos^2\left(\frac{A}{2}\right) = \frac{1}{2}[1 + \cos(2)]$
$\tan^2\left(\frac{A}{2}\right) = \frac{[1 - \cos(A)]}{[1 + \cos(A)]}$
$\tan^2\left(\frac{A}{2}\right) = \left[\frac{\sin(A)}{1 + \cos(A)}\right]^2$
$\tan^2\left(\frac{A}{2}\right) = \left[\frac{1 - \cos^2(A)}{\sin(A)}\right]^2$

Inverse Trigonometric Functions
Inverse trigonometric functions require us to restrict the domain of regular trigonometric functions as follows.

Trigonometric Function	**Restricted Domain**
Sine and Cosecant	$[-90^\circ, 90^\circ]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Cosine and Secant	$[0^\circ, 180^\circ]$ or $[0, \pi]$
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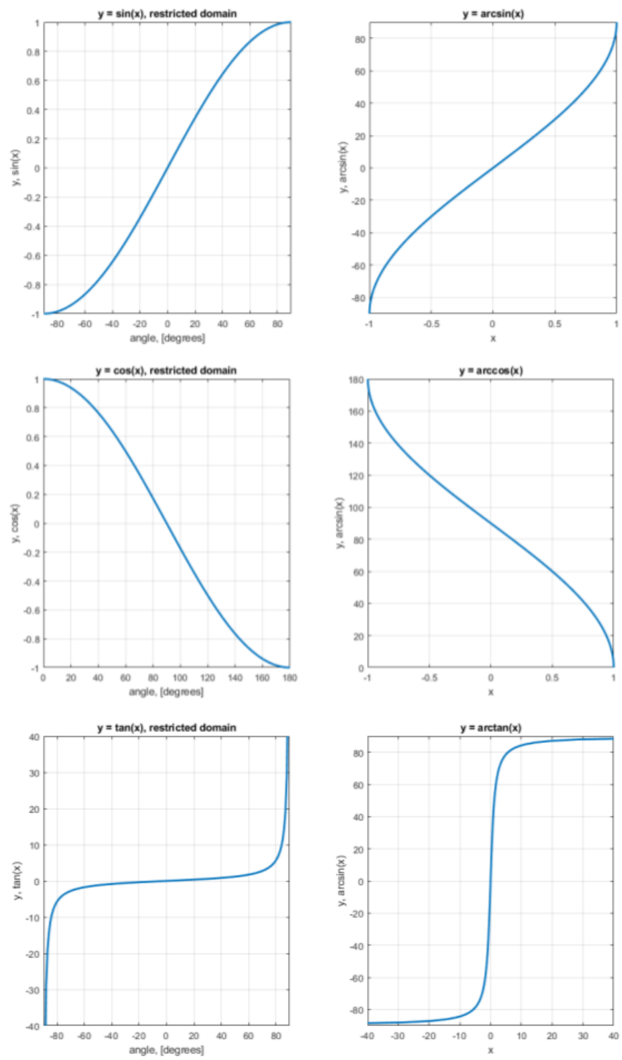

By: ferrantetutoring