Pre-Calculus Exponentials and Logarithms Review

Exponential and logarithmic functions are used in a variety of practical applications and are also some of the central functions we will study in calculus. For these reasons we provide this review.

Exponential functions are of the form:

$$f(x) = B^x$$

Where, B is called the base and has the following restrictions: $B > 0, B \neq 1$

A logarithmic function with a base, B, is the inverse of the exponential function with the same base.

$$g_B(x) = {f_B}^{-1}(x)$$

The notation used for a logarithmic function is as follows:

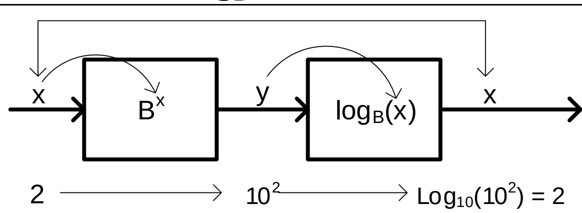
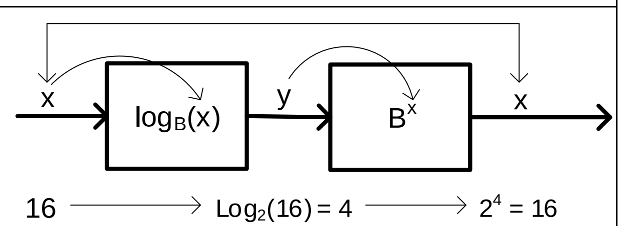
$$f(x) = \log_B(x)$$

Where, $B > 0$ and $B \neq 1$.

The definition of a logarithmic function is usually given in terms of its inverse and can be stated as follows:

Given $y = \log_B(x)$: y is a number such that when we raise the base, B, to the power of y the result is x. $If\!: \{y = \log_B(x)\},\ Then\!: \{B^y = x\}$

Recall inverse functions are such that if a function and its inverse are performed one after the other, other, i.e. composed, they will "undo" the effects of each other. This is shown below in equation form and function machine form along with a simple example for each case.

$\boldsymbol{\log_B(B^x) = x}$	$\boldsymbol{B^{\log_B(x)} = x}$
	

Graphing Exponential Functions:

The two types of behavior we can have with exponential functions are growth and decay. Exponential growth occurs when $B > 1$, and exponential decay occurs for $0 < B < 1$. To help us plot exponential functions we must recall the following fundamental property of exponents.

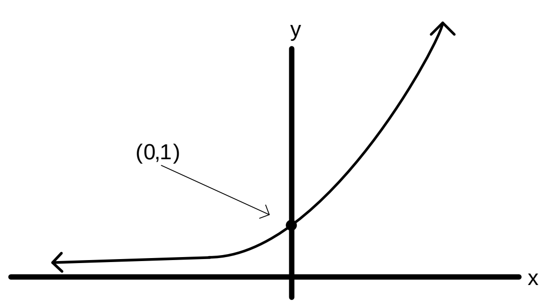
$$\frac{1}{B^x} = B^{-x}$$

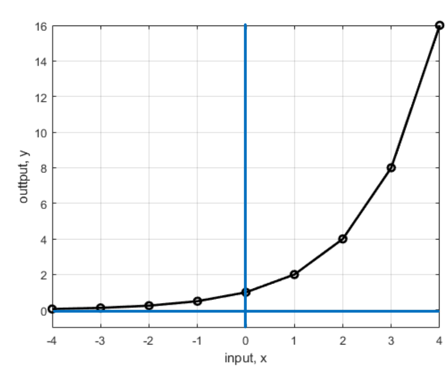
Exponential Growth: $f(x) = B^x, B > 1$

To help us get an intuitive feel let's start the graph by examining the behavior as x approaches negative and positive infinity. We can use $B = 2$ as an example. From the table below, we see that as x goes to negative infinity $f(x)$ gets smaller and smaller but will never reach zero. Additionally, as x goes to positive infinity $f(x)$ growths without bound. Finally, we note that when $x = 0$, $f(x) = 1$. This is true for any base since $B^0 = 1$.

Let: $x \to -\infty$		**Let: $x \to +\infty$**	
x	2^x	x	2^x
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	1	$2^1 = 2$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	2	$2^2 = 4$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	3	$2^3 = 8$
⋮	⋮	⋮	⋮
-10	$2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$	10	$2^{10} = 1024$

Based on the above observations we can create a rough sketch of an exponential growth function as shown in the figure on the left below. The figure on the right is a precise graph of the function $f(x) = 2^x$



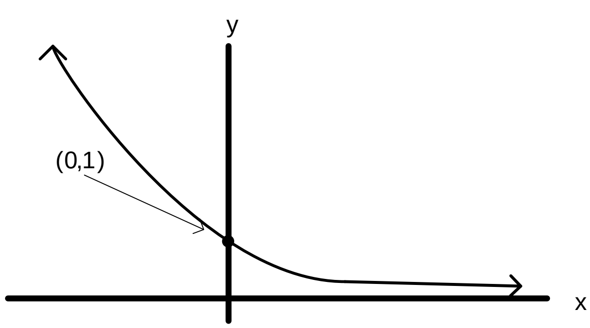


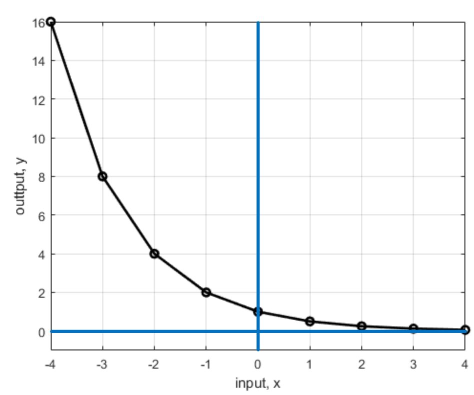
Exponential Decay: $f(x) = B^x, 0 < B < 1$

We again start by examining the behavior as x approaches negative and positive infinity. In this case we use $B = \frac{1}{2}$. From the table we can see the opposite behavior from the exponential growth case; i.e. as x goes to negative infinity $f(x)$ grows without bound, and as x goes to positive infinity $f(x)$ gets smaller and smaller without ever reaching zero. Conversely, we see the same behavior at $x = 0$; namely $f(x) = 1$.

Let: $x \to -\infty$		**Let: $x \to +\infty$**	
x	$\left(\frac{1}{2}\right)^x$	x	$\left(\frac{1}{2}\right)^x$
-1	$\left(\frac{1}{2}\right)^{-1} = \frac{1}{2^{-1}} = 2^1 = 2$	1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2^1} = \frac{1}{2}$
-2	$\left(\frac{1}{2}\right)^{-2} = \frac{1}{2^{-2}} = 2^2 = 4$	2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{2^2} = \frac{1}{4}$
-3	$\left(\frac{1}{2}\right)^{-3} = \frac{1}{2^{-3}} = 2^3 = 8$	3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{2^3} = \frac{1}{8}$
⋮	⋮	⋮	⋮
-10	$\left(\frac{1}{2}\right)^{-10} = \frac{1}{2^{-10}} = 2^{10} = 1024$	10	$\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = \frac{1}{1024}$

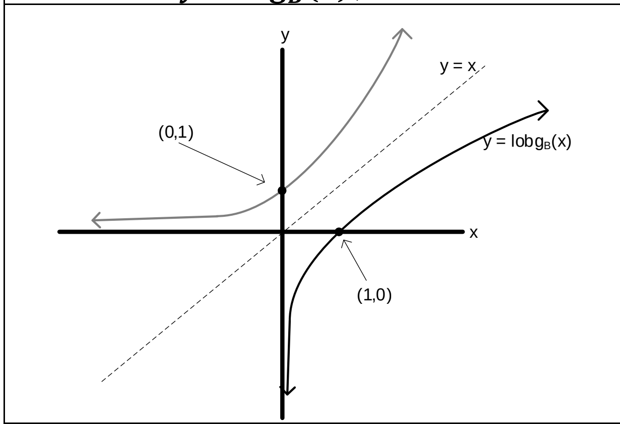
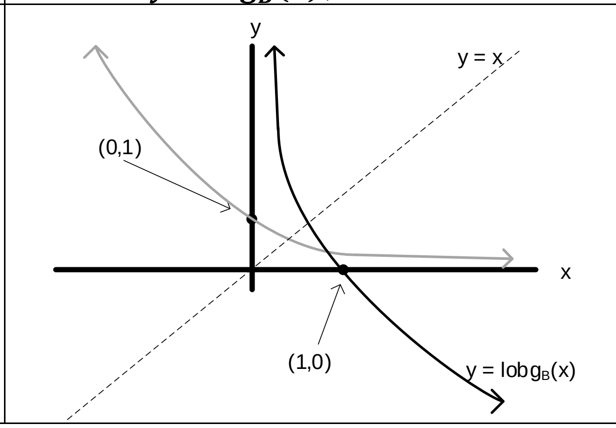
We can again create a rough sketch of an exponential decay function as well as a precise graph of $f(x) = \left(\frac{1}{2}\right)^x$, both of which are shown below.





Graphing Logarithmic Functions:

Logarithmic functions also have different forms for $B > 1$, and for $0 < B < 1$. One way to get an intuitive feel for the graph of logarithmic functions is to recall that the graph of an inverse function reflects the graph of the original function across the line $y = x$. Recall also that each coordinate of the original function, (x, y), corresponds to a coordinate of the inverse function if we switch the order of coordinates, i.e. (y, x). The sketch below shows this by mapping the point $(0,1)$ from the exponential function to the point $(1,0)$ on the logarithmic function.

$y = \log_B(x)\,, B > 1$	$y = \log_B(x)\,, 0 < B < 1$
	

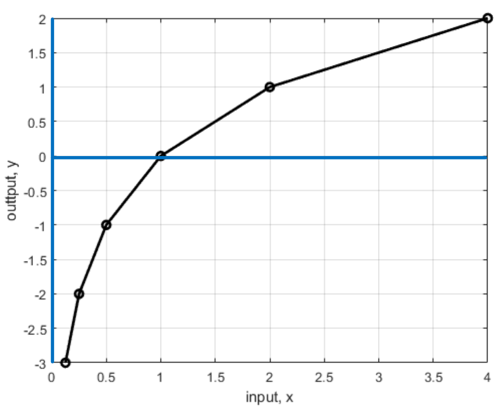
This switching of coordinates can also be stated more generally as switching the domain and range.

$f(x) = B^x$		$f(x) = \log_B(x)$	
Domain	***Range***	***Domain***	***Range***
$(-\infty, \infty)$	$(0, \infty)$	$(0, \infty)$	$(-\infty, \infty)$

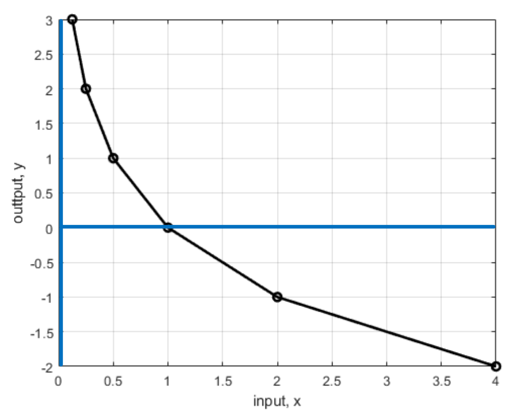
Of course, we can also create more precise plots using a table as we did for the exponential graphs above. The table below uses both $B = 2$, and $B = 1/2$. The graphs are also shown for illustration.

x	$\log_2(x)$	$\log_{1/2}(x)$
$\frac{1}{8}$	$\log_2\left(\frac{1}{8}\right) = -3$ Since: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$\log_{1/2}\left(\frac{1}{8}\right) = 3$ Since: $\left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}$
$\frac{1}{4}$	$\log_2\left(\frac{1}{4}\right) = -2$ Since: $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\log_{1/2}\left(\frac{1}{4}\right) = 2$ Since: $\left(\frac{1}{2}\right)^2 = \frac{1}{2^2} = \frac{1}{4}$
$\frac{1}{2}$	$\log_2\left(\frac{1}{2}\right) = -1$ Since: $2^{-1} = \frac{1}{2^1} = \frac{1}{1}$	$\log_{1/2}\left(\frac{1}{2}\right) = 1$ Since: $\left(\frac{1}{2}\right)^1 = \frac{1}{2^1} = \frac{1}{2}$
1	$\log_2(1) = 0$ Since: $2^0 = 1$	$\log_{1/2}(1) = 0$ Since: $\left(\frac{1}{2}\right)^0 = 1$
2	$\log_2(2) = 1$ Since: $2^1 = 2$	$\log_{1/2}(2) = -1$ Since: $\left(\frac{1}{2}\right)^{-1} = \frac{1}{2^{-1}} = 2^1 = 2$
4	$\log_2(4) = 2$ Since: $2^2 = 4$	$\log_{1/2}(4) = -2$ Since: $\left(\frac{1}{2}\right)^{-2} = \frac{1}{2^{-2}} = 2^2 = 4$

$y = \log_2(x)$



$y = \log_{1/2}(x)$



Exponential and Logarithmic Function Transformations:

The function transformation we studied earlier can of course be applied to exponential and logarithmic functions. As a reminder we show a few examples using the exponential functions.

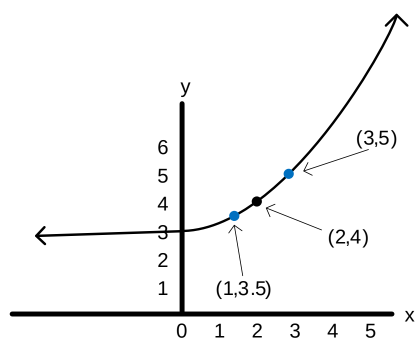
Example 1: Graph $\boldsymbol{f(x) = 2^{(x-2)} + 3}$

The parent function is $p(x) = 2^x$, and the transformations are as follows:

1. Shift right by 2
2. Shift up by 3

Using this knowledge, we can map the $(0, 1)$ point from the parent graph to the point $(0 + 2, 1 + 3) = (2,4)$ on the transformed graph. As there are no reflections, we already have enough information to sketch the graph. To make the graph more precise we can choose one x value on both sides of the point $(2, 4)$ to plot.

$$f(1) = 2^{(1-2)} + 3 = \frac{1}{2} + 3 = 3.5$$
$$f(3) = 2^{(3-2)} + 3 = 2 + 3 = 5$$



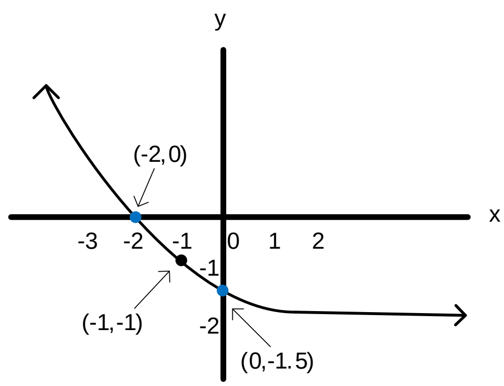
Example 2: Graph $\boldsymbol{f(x) = \left(\frac{1}{2}\right)^{(x+1)} - 2}$

The parent function is $p(x) = \left(\frac{1}{2}\right)^x$, and the transformations are as follows:

3. Shift left by 1
4. Shift down by 2

Using this knowledge, we can map the $(0, 1)$ point from the parent graph to the point $(0 - 1, 1 - 2) = (-1, -1)$ on the transformed graph. As there are no reflections, we already have enough information to sketch the graph. To make the graph more precise we can choose one x value on both sides of the point $(-1, -1)$ to plot.

$$f(-2) = \left(\frac{1}{2}\right)^{(-2+1)} - 2 = 2 - 2 = 0$$
$$f(0) = \left(\frac{1}{2}\right)^{(0+1)} - 2 = \frac{1}{2} - 2 = -1.5$$



Example 2: Graph $\boldsymbol{f(x) = -2^{(-x+1)} + 3}$

We should first rewrite the function as follows:

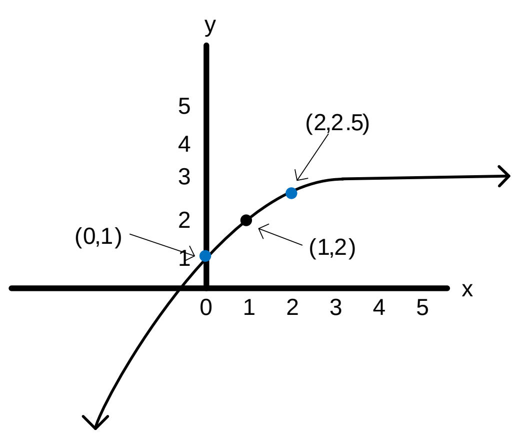
$$f(x) = \left(-2^{-(x-1)}\right) + 3$$

The parent function is $p(x) = 2^x$. In this case we list the transformations as we map the point $(0, 1)$ through these transformations.

1. Shift right by 1	1. $(0,1) \to (1,1)$
2. Reflect across $y = 1$ axis	2. $(1,1) \to (1,1)$
3. Reflect across x axis	3. $(1,1) \to (1,-1)$
4. Shift up by 3	4. $(1,-1) \to (1,2)$

Based on this point and the reflections, we already have enough information to sketch the graph. And as before to make the graph more precise we can choose one x value on both sides of the point $(1, 2)$ to plot.

$$f(0) = -2^{(-0+1)} + 3 = -2 + 3 = 1$$
$$f(2) = -2^{(-2+1)} + 3 = -\frac{1}{2} + 3 = 2.5$$



The Number e and the Natural Logarithm:

The number e is just as important as the number π in mathematics. There are various ways to define the number π, but the most common is that it is the ratio of the circumference to the diameter of a circle.

$$\pi = \frac{C}{D}$$

Similarly, the number e has various definitions, but unfortunately most require a knowledge of calculus to fully appreciate. One definition however can be obtained through the study of compound interest, which we will show in the application section. For now, we simply state that the number e is an irrational number like π whose value is approximately $e \approx 2.718$. The number is sometimes called Euler's number after the Swiss mathematician Leonhard Euler. When this number is used as the base of an exponential function, $B = e$, we call it the *natural* base.

$$f(x) = e^x$$

Similarly, the logarithm function that uses this base is called the natural logarithm and has a special notation shown below.

$$f(x) = \log_e(x) \stackrel{\text{def}}{=} \ln(x)$$

Computations involving the natural base exponential function, e^x, and the natural logaritghm function, $\ln(x)$, are treated just like computations with ordinary exponential and logarithm functions, however, as we shall soon see, these functions will indeed play a fundamental role in our study of calculus.

Exponential and Logarithmic Properties:

When working with equations involving exponential and/or logarithmic functions it helps to keep in mind various properties. We start by defining the inverse rule for both exponentials and logarithms. The inverse rule is a key rule used to solve exponential and logarithmic equations. It follows directly from the definition of inverse functions and was shown in the first section using function machines to illustrate.

Solve for x in a logarithm function by exponentiating	Solve for x in an exponential function by taking the logarithm
$\boldsymbol{B^{\log_B(x)} = x}$	$\boldsymbol{\log_B(B^x) = x}$

Exponent Rules:

1. *Zero Exponent Rule:* Any number raised to the power zero is one.

$$\boldsymbol{B^0 = 1}$$

2. *Product and Quotient Rule:* Multiplying/dividing exponentiated numbers with the same base is the same as adding/subtracting the exponents.

$\boldsymbol{B^x \cdot B^y = B^{(x+y)}}$	$\boldsymbol{\frac{B^x}{B^y} = B^{(x-y)}}$
<u>*Example:*</u> $B^3 \cdot B^2 = B^5$ $(B \cdot B \cdot B) \cdot (B \cdot B) = B^5$ $(B \cdot B \cdot B \cdot B \cdot B) = B^5$	<u>*Example:*</u> $\frac{B^5}{B^3} = B^2$ $\frac{(B \cdot B \cdot \cancel{B \cdot B \cdot B})}{(\cancel{B \cdot B \cdot B})} = B^2$ $(B \cdot B) = B^2$

3. *Negative Exponent Rule:* An exponentiated number can be moved from the numerator to the denominator, or denominator to numerator, by changing the sign of the exponent. This property can be shown in various forms as shown below.

$\boldsymbol{B^{-x} = \frac{1}{B^x}}$	$\boldsymbol{B^x = \frac{1}{B^{-x}}}$	$\boldsymbol{\frac{A^x}{B^y} = \frac{B^{-y}}{A^{-x}}}$	$\boldsymbol{\left(\frac{A}{B}\right)^{-x} = \left(\frac{B}{A}\right)^x}$

This property is easily shown using knowledge of the properties above.

$$B^x = \frac{B^x}{1} \cdot \left(\frac{B^{-x}}{B^{-x}}\right) = \frac{B^{x-x}}{B^{-x}} = \frac{B^0}{B^{-x}} = \frac{1}{B^{-x}}$$

4. *Exponent Power Rules:*

$\boldsymbol{(B^x)^y = B^{x \cdot y}}$	$\boldsymbol{(A \cdot B)^x = A^x B^x}$	$\boldsymbol{\left(\frac{A}{B}\right)^x = \frac{A^x}{B^x}}$
<u>*Example:*</u> $(B^3)^2 = B^6$ $(B^3) \cdot (B^3) = B^6$ $(B \cdot B \cdot B) \cdot (B \cdot B \cdot B) = B^6$ $(B \cdot B \cdot B \cdot B \cdot B \cdot B) = B^6$	<u>*Example:*</u> $(A \cdot B)^3$ $= A^3 B^3$ $(AB) \cdot (AB) \cdot (AB)$ $= A^3 B^3$ $(A \cdot A \cdot A) \cdot (B \cdot B \cdot B)$ $= A^3 B^3$	<u>*Example:*</u> $\left(\frac{A}{B}\right)^3 = \frac{A^3}{B^3}$ $\left(\frac{A}{B}\right) \cdot \left(\frac{A}{B}\right) \cdot \left(\frac{A}{B}\right) = \frac{A^3}{B^3}$ $\frac{(A \cdot A \cdot A)}{(B \cdot B \cdot B)} = \frac{A^3}{B^3}$

5. *Exponent Root Rule:*

$\sqrt[y]{\boldsymbol{B}^{\boldsymbol{x}}} = \boldsymbol{B}^{\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right)} = \left(\sqrt[y]{\boldsymbol{B}}\right)^{\boldsymbol{x}}$
<u>*Example:*</u> $\sqrt[2]{2^4} = 2^{\left(\frac{4}{2}\right)} = \left(\sqrt[2]{2}\right)^4$ $\sqrt[2]{16} = 2^{(2)} = \left(\sqrt[2]{2}\cdot\sqrt[2]{2}\cdot\sqrt[2]{2}\cdot\sqrt[2]{2}\right)$ $4 = 4 = (2\cdot 2)$ $4 = 4 = 4$

Logarithm Rules:

The logarithmic properties are based on the corresponding properties of exponents.

1. *Zero Logarithm Rule:* The logarithm of one is zero.

$$\mathbf{log}_{\boldsymbol{B}}(\mathbf{1}) = \mathbf{0}$$

2. *Product and Quotient Rule*: The logarithm of a product/quotient is equivalent to the sum/difference of the logarithms of each value.

$\mathbf{log}_{\boldsymbol{B}}(\boldsymbol{xy}) = \mathbf{log}_{\boldsymbol{B}}(\boldsymbol{x}) + \mathbf{log}_{\boldsymbol{B}}(\boldsymbol{y})$	$\mathbf{log}_{\boldsymbol{B}}\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right) = \mathbf{log}_{\boldsymbol{B}}(\boldsymbol{x}) - \mathbf{log}_{\boldsymbol{B}}(\boldsymbol{y})$

3. *Reciprocal Rule:* This rule follows directly from the quotient rule.

$$\log_B\left(\frac{1}{y}\right) = \log_B(1) - \log_B(y)$$

$$\log_B\left(\frac{1}{y}\right) = 0 - \log_B(y)$$

$$\mathbf{log}_{\boldsymbol{B}}\left(\frac{\mathbf{1}}{\boldsymbol{y}}\right) = -\mathbf{log}_{\boldsymbol{B}}(\boldsymbol{y})$$

4. *Exponent Rule:* The logarithm of an exponentiated number is equivalent to the exponent multiplied by the logarithm of the base of the number.

$$\mathbf{log}_{\boldsymbol{B}}(\boldsymbol{x}^{\boldsymbol{y}}) = \boldsymbol{y}\,\mathbf{log}_{\boldsymbol{B}}(\boldsymbol{x})$$

Finally, we review derive the formulas required to change the base of exponents and logarithms. For various reasons, some of which we will see later in our study of calculus, it is sometimes most convenient to represent all exponentials and logarithmic functions using the natural base, e. Below we derive general, i.e. change from any base, B of any other base, A, *change of base* formulas for exponentials and logarithms.

Exponential Base Change:

The inverse rule provides us with a method to write any number, A, as follows:

$$A = B^{\log_B(A)}$$

Raising both sides to the power of x and using the exponent power rule we have a change of base formula from base A to base B.

$$A^x = \left(B^{\log_B(A)}\right)^x$$

$$\boldsymbol{A^x = B^{(x \log_B(A))}}$$

Changing to the natural base, e, we have the following:

$$\boxed{\boldsymbol{A^x = e^{x \ln(A)}}}$$

Logarithmic Base Change:

To derive the logarithmic change of base formula we start with the basic definition of the logarithm as we defined in the first section.

$$If\colon \{y = \log_A(x)\}, \qquad Then\colon \{A^y = x\}$$

Taking the base B logarithm of both sides of the "*Then*" equation we can now derive the logarithmic change of base formula.

$$\begin{aligned} \log_B(A^y) &= \log_B(x) \\ y \log_B(A) &= \log_B(x) \\ \frac{y \cancel{\log_B(A)}}{\cancel{\log_B(A)}} &= \frac{\log_B(x)}{\log_B(A)} \\ y &= \frac{\log_B(x)}{\log_B(A)} \end{aligned}$$

And since $y = \log_A(x)$ we have the final change of base formula.

$$\boldsymbol{\log_A(x) = \frac{\log_B(x)}{\log_B(A)}}$$

Again, changing to the natural logarithm, $\ln(\cdot)$, we have the following:

$$\boxed{\boldsymbol{\log_A(x) = \frac{\ln(x)}{\ln(A)}}}$$

Solving Exponential and Logarithmic Equations:

We can use the properties from the previous section to solve exponential and logarithmic equations as shown below.

Example 1: Solve the following exponential equation for x

$$3^{2x-1} = 5^x$$

We start by taking the natural logarithm, (note we can use any base), of both sides.

$$\begin{aligned} \ln(3^{2x-1}) &= \ln(5^x) \\ (2x-1)\cdot\ln(3) &= x\ln(5) \\ 2x\ln(3) - \ln(3) &= x\ln(5) \\ x(2\ln(3) - \ln(5)) &= \ln(3) \\ x &= \frac{\ln(3)}{(2\ln(3) - \ln(5))} \\ x &= 1.896 \end{aligned}$$

Example 2: Solve the following exponential equation for x

$$2e^x = 3 - e^{-x}$$

In this case we start by setting the equation to zero and multiplying through by e^x.

$$2e^x - 3 + \frac{1}{e^x} = 0$$

$$e^x\left(2e^x - 3 + \frac{1}{e^x} = 0\right)$$

$$2(e^x)^2 - 3(e^x) + 1 = 0$$

If we temporarily substitute $y = e^x$, we can solve the resulting quadratic formula by factoring.

$$\begin{aligned} 2y^2 - 3y + 1 = 0 \\ (2y-1)(y-1) = 0 \end{aligned}$$

Therefore, we have two possible solutions: $y = \frac{1}{2}$ and $y = 1$

Resubstituting and solving for x we now have:

$$\begin{aligned} e^x &= 1/2 \\ x &= \ln(1/2) \end{aligned}$$

$$\begin{aligned} e^x &= 1 \\ x &= \ln(1) \\ x &= 0 \end{aligned}$$

Finally, we substitute back into the original equation to verify that both solution work.

$$\begin{aligned} 2e^{\ln\left(\frac{1}{2}\right)} &= 3 - e^{-\ln\left(\frac{1}{2}\right)} \\ 2\cdot\left(\frac{1}{2}\right) &= 3 - e^{\ln(2)} \\ 1 &= 3 - 2 \\ 1 &= 1 \end{aligned}$$

$$\begin{aligned} 2e^0 &= 3 - e^{-0} \\ 2\cdot 1 &= 3 - 1 \\ 2 &= 2 \end{aligned}$$

Example 3: Solve the following logarithmic equation for x

$$\log_3(x+14) - \log_3(x+6) = \log_3(x)$$

We start by using the quotient rule for logarithms and raising both sides to the power of 3.

$$\log_3\left(\frac{x+14}{x+6}\right) = \log_3(x)$$
$$3^{\log_3\left(\frac{x+14}{x+6}\right)} = 3^{\log_3(x)}$$
$$\left(\frac{x+14}{x+6}\right) = x$$
$$x+14 = x^2+6x$$
$$x^2+5x-14 = 0$$
$$(x+7)(x-2) = 0$$

Which gives us two possible solutions: $x = -7$, and $x = 2$, which we substitute back to verify.

$\log_3(-7+14) - \log_3(-7+6) = \log_3(-7)$ $\log_3(7) - \log_3(-1) = \log_3(-7)$ Since the logarithm of a negative number is undefined $x = -7$ is not a solution.	$\log_3(2+14) - \log_3(2+6) = \log_3(2)$ $\log_3\left(\frac{16}{8}\right) = \log_3(2)$ $\log_3(2) = \log_3(2)$ Therefore, $x = 2$ is a solution

Example 4: Solve the following logarithmic equation for x

$$\sqrt{\ln(x)} = \ln\left(\sqrt{x}\right)$$

We can start by rewriting the right-hand side as follows:

$$\sqrt{\ln(x)} = \ln\left((x)^{\frac{1}{2}}\right)$$
$$\sqrt{\ln(x)} = \frac{1}{2}\ln(x)$$

Squaring both side and letting $\ln(x) = y$ we have.

$$\ln(x) = \frac{1}{4}(\ln(x))^2$$
$$y = \frac{1}{4}(y)^2$$
$$\frac{1}{4}y^2 - y = 0$$
$$y\left(\frac{1}{4}y - 1\right) = 0$$

Which gives us two possible solutions: $y = 0$, and $y = 4$.

Finally, we substitute back to solve for x and check the solutions in the original equation.

$\ln(x) = 0$ $x = 1$	$\ln(x) = 4$ $x = e^4$
$\sqrt{\ln(1)} = \ln(\sqrt{1})$ $0 = 0$	$\sqrt{\ln(e^4)} = \ln\left(\sqrt{e^4}\right)$ $\sqrt{4} = \ln(e^2)$ $2 = 2$

Application: Compound Interest

An important and very practical application of exponential functions is bank interest. Let's look at the per year balance when you initially deposit money into an interest-bearing savings account that is compounded yearly. The balance at time t is denoted as $B(t)$, where t is measured in years. The interest rate is given in decimal form as r, e.g. for 5% interest $r = 0.05$, and is assumed to be compounded yearly. Finally, the initial deposit is the balance at $t = 0$, $B(0)$. Let's see if we can derive a formula for the balance, $B(t)$, at any time t using the table below.

time, $\boldsymbol{t}$	Balance, $\boldsymbol{B(t)}$
1	$B(1) = B(0) + B(0)r$ $\boldsymbol{B(1) = B(0)(1 + r)}$
2	$B(2) = B(1)(1 + r)$ $B(2) = B(0)(1 + r)(1 + r)$ $\boldsymbol{B(2) = B(0)(1 + r)^2}$
3	$B(3) = B(2)(1 + r)$ $B(3) = B(0)(1 + r)^2(1 + r)$ $\boldsymbol{B(3) = B(0)(1 + r)^3}$
4	$B(4) = B(3)(1 + r)$ $B(4) = B(0)(1 + r)^3(1 + r)$ $\boldsymbol{B(4) = B(0)(1 + r)^4}$
. . .	. . .
t	$\boldsymbol{B(t) = B(0)(1 + r)^t}$

Therefore, we can find the balance at the end of t years with the following exponential function.

$$B(t) = B(0)(1 + r)^t$$

As an example, assume you initially deposit $10,000 into a savings account giving a 7% yearly interest rate that is compounded yearly. What is the amount after 5 years? How many years does it take for your initial deposit to double?

The balance after 5 years is found as follows:

$$B(5) = \$10{,}000(1 + 0.07)^5$$
$$B(5) = \$14{,}025.51$$

When the initial deposit doubles you will have $\$2B(0)$, therefore we have:

$$2\cancel{B(0)} = \cancel{B(0)}(1 + r)^t$$
$$2 = (1 + r)^t$$
$$\ln(2) = t\ln(1 + r)$$
$$t = \frac{\ln(2)}{\ln(1.07)}$$
$$t = 10.24 \ \rightarrow 11 \ years$$

Next we extend this equation to handle cases when the compounding takes place more frequent, e.g. monthly, weekly, daily, continuously.

If we let N represent the number of times the interest is compounded per year, then the yearly interest rate applied at each compounding instance is given as r/N. Furthermore, we multiply the variable t, which denotes a year, in the equation by N since at the end of each year the interest would have been compounded Nt times. With this we can rewrite our equation as follows:

$$B(t) = B(0)\left(1 + \frac{r}{N}\right)^{Nt}$$

Where, N is the number of compounding instances per year.

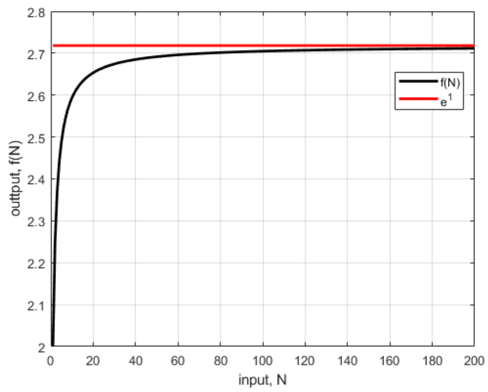
Let's find the account balance after 5 years using the same example as above, but with the interest being compounded quarterly, i.e. 4 times per year.

$$B(t) = \$10{,}000\left(1 + \frac{0.07}{4}\right)^{4\cdot 5}$$
$$B(t) = \$14{,}190.20$$

Finally, let's see what happens if we let the interest get compounded many, many times a year, i.e. we let N get very large. Using the power property of exponents, we first rewrite the equation from above as follows.

$$B(t) = B(0)\left[\left(1 + \frac{r}{N}\right)^N\right]^t$$

For the moment we let $r = 1$ and plot only the term in brackets as a function of N. We also plot the value of e^1.



$$f(N) = \left(1 + \frac{1}{N}\right)^N$$

Interestingly, we see that as N gets larger $\left(1 + \frac{1}{N}\right)^N$ seems to approach the value, e^1! This is indeed true, and as it turns out if we use values of r other than one we will see that $f(N)$ approaches e^r. Even though we haven't formally defined the notion of a limit we give the following definition of e^x below.

In the limit as N approaches infinity the value of $\left(1 + \frac{x}{N}\right)^N$ is equal to e^x for any value of x. $$e^x = \lim_{N \to \infty} \left\{\left(1 + \frac{x}{N}\right)^N\right\}$$

When we let N get very large in our bank balance equation we say the interest is *compounded continuously,* and we can replace the $\left(1 + \frac{r}{N}\right)^N$ term with e^r. Therefore, the balance in an account at any time, t, where the interest is compounded continuously is given as follows.

$$B(t) = B(0)[e^r]^t$$

$$B(t) = B(0)e^{rt}$$

Finally, let's use the same example from above to find the balance after 5 years and the time when the account doubles. The balance is given as follows:

Balance	*Time to double*
$B(5) = \$10{,}000e^{0.07 \cdot 5}$ $B(5) = \$14{,}190.68$	$2\cancel{B(0)} = \cancel{B(0)}e^{rt}$ $\ln(2) = rt$ $t = \frac{\ln(2)}{0.07}$ $t = 9.9 \rightarrow 10\ years$

Final Summary for Pre-Calc Exponential and Logarithm Review

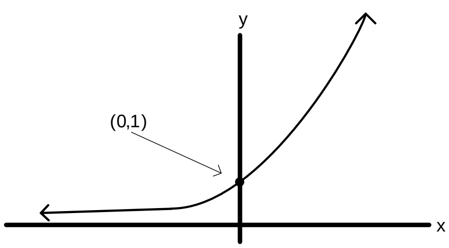
Exponential Function

Exponential functions are of the form:

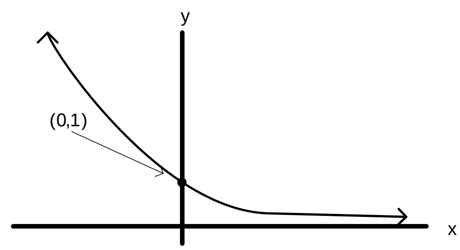
$$f(x) = B^x$$

Where, B is called the base and has the following restrictions: $B > 0, B \neq 1$
Domain:$(-\infty, \infty)$, Range: $(0, \infty)$

Exponential Growth, $B > 1$



Exponential Decay, $B < 1$



Logarithmic Function

A logarithmic function with a base, B, is the inverse of the exponential function with the same base.

$$g_B(x) = {f_B}^{-1}(x)$$

The notation used for a logarithmic function is as follows:

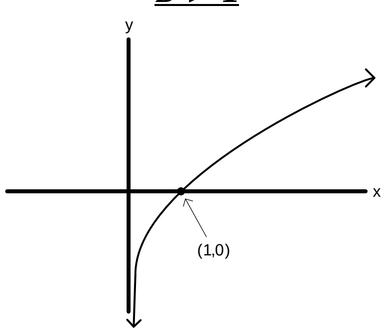
$$f(x) = \log_B(x)$$

Where, $B > 0$ and $B \neq 1$, Domain:$(0, \infty)$, Range: $(-\infty, \infty)$

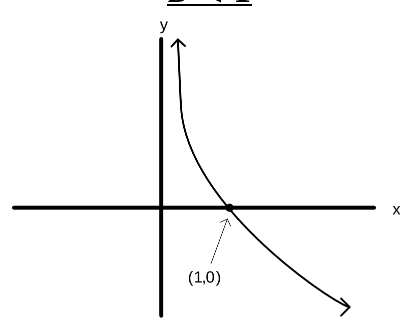
Given $y = \log_B(x)$: y is a number such that when we raise the base, B, to the power of y the result is x.

$$If\text{: } \{y = \log_B(x)\}, \qquad Then\text{: } \{B^y = x\}$$

$B > 1$



$B < 1$



The Number e and the Natural Logarithm

Euler's number e is an irrational number and is fundamental to mathematics with various definitions and applications. Once such definition is as follows:

$$e^x = \lim_{N\to\infty} \left\{ \left(1 + \frac{x}{N}\right)^N \right\}$$

And $e^1 \approx 2.718$

When used as the base in a logarithmic function it is called the natural logarithm and is given a special notation.

$$\log_e(x) \stackrel{\text{def}}{=} \ln(x)$$

Exponent Rules

Exponent Rules:

1. *Zero Exponent Rule:* Any number raised to the power zero is one.

$$\boldsymbol{B^0 = 1}$$

2. *Product and Quotient Rule:* Multiplying/dividing exponentiated numbers with the same base is the same as adding/subtracting the exponents.

$$\boldsymbol{B^x \cdot B^y = B^{(x+y)}} \qquad \boldsymbol{\frac{B^x}{B^y} = B^{(x-y)}}$$

3. *Negative Exponent Rule:* An exponentiated number can be moved from the numerator to the denominator, or denominator to numerator, by changing the sign of the exponent. This property can be shown in various forms as shown below.

$$\boldsymbol{B^{-x} = \frac{1}{B^x}} \qquad \boldsymbol{B^x = \frac{1}{B^{-x}}} \qquad \boldsymbol{\frac{A^x}{B^y} = \frac{B^{-y}}{A^{-x}}} \qquad \boldsymbol{\left(\frac{A}{B}\right)^{-x} = \left(\frac{B}{A}\right)^x}$$

4. *Exponent Power Rules:*

$$\boldsymbol{(B^x)^y = B^{x \cdot y}} \qquad \boldsymbol{(A \cdot B)^x = A^x B^x} \qquad \boldsymbol{\left(\frac{A}{B}\right)^x = \frac{A^x}{B^x}}$$

5. *Exponent Root Rule:*

$$\boldsymbol{\sqrt[y]{B^x} = B^{\left(\frac{x}{y}\right)} = \left(\sqrt[y]{B}\right)^x}$$

6. *Exponent Base Change:*

$$\boldsymbol{A^x = B^{(x \log_B(A))}}$$

Changing to the natural base, e, we have the following:

$$\boldsymbol{A^x = e^{x \ln(A)}}$$

Logarithm Rules
1. *Zero Logarithm Rule:* The logarithm of one is zero. $$\log_B(1) = 0$$ 2. *Product and Quotient Rule*: The logarithm of a product/quotient is equivalent to the sum/difference of the logarithms of each value. $$\log_B(xy) = \log_B(x) + \log_B(y) \qquad \log_B\left(\frac{x}{y}\right) = \log_B(x) - \log_B(y)$$ 3. *Reciprocal Rule:* This rule follows directly from the quotient rule. $$\log_B\left(\frac{1}{y}\right) = -\log_B(y)$$ 4. *Exponent Rule:* The logarithm of an exponentiated number is equivalent to the exponent multiplied by the logarithm of the base of the number. $$\log_B(x^y) = y\log_B(x)$$ 5. *Logarithm Base Change:* $$\log_A(x) = \frac{\log_B(x)}{\log_B(A)}$$ Changing to the natural logarithm, we have the following: $$\log_A(x) = \frac{\ln(x)}{\ln(A)}$$
Compound Interest
The balance at any time t, $B(t)$, in an interest-bearing account where the interest is ***compounded N times per year*** at a yearly interest rate of $R\%$ is given by: $$B(t) = B(0)\left(1 + \frac{r}{N}\right)^{Nt}$$ Where, t is the number of years, $B(0)$ is the initial deposit, and $r = R/100$. When the interest in ***compounded continuously***, *i.e.* $N \to \infty$, the balance can be found as follows: $$B(t) = B(0)e^{rt}$$

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