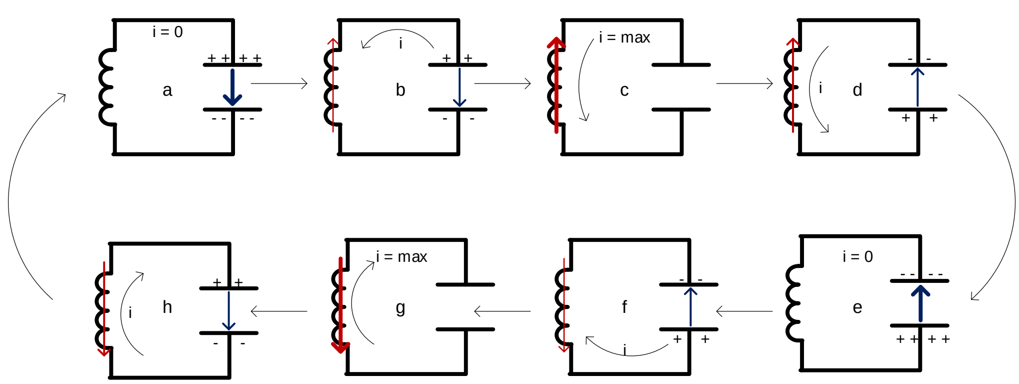
RLC Circuits and Alternating Current (AC) Introduction

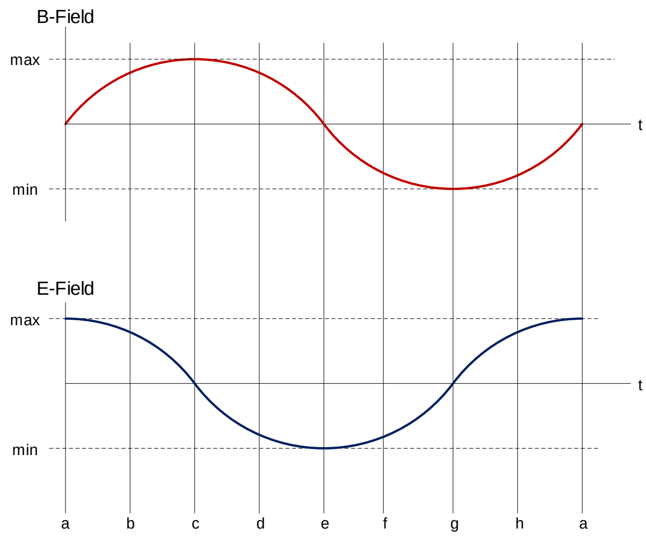
LC Circuits

The three basic electrical components we studied are the resistor, the capacitor, and the inductor. We have so far examined RC and RL circuits, where we have seen that the current and voltage grow and decay exponentially. We now combine the capacitor and inductor to create an LC circuit, where we will find that the current and voltage instead will vary sinusoidally in time. Let's begin with a qualitative look at an LC circuit.

The first figure below shows an RL circuit at various snapshots of time (i.e. t = a, b, c. …). We assume that at $t < 0$ the capacitor is charged, and the circuit is open. Then at $t = 0$, the circuit is closed. The blue arrow indicates the electric field and the red arrows indicate the magnetic field, with the thickness of the arrows indicating the magnitude of the fields. The second figure shows a plot of the field strengths over time. Below we give a brief description for each snapshot of time.

a) The capacitor starts fully charged so the E-field is at its maximum value. As current has not started flowing yet, the B-field is zero.
b) The capacitor begins to discharge through the inductor, so the E-field begins to get smaller while the B-field grows. Current flows in a counter-clockwise direction.
c) The capacitor is fully discharged, and the maximum current is flowing in the circuit. At this instant the E-field is zero and the B-field is at its maximum value.
d) The capacitor now begins to charge with positive charge building on the lower plate now.
e) The capacitor is again fully charged as it was at time a., but the E-field is pointing in the opposite direction. The B-field is again zero.
f) The capacitor now begins to discharge again, but the current is now flowing in the clockwise direction. The B-filed also start to grow in the opposite direction as it did at time b.
g) The capacitor is again fully discharged, and the maximum current is again flowing in the circuit clockwise. At this instant the E-field is zero and the B-filed is at its maximum value.
h) The capacitor now begins to charge again with the same polarity that is started with. When the capacitor becomes fully charged, we are back at time a. and the entire process repeats.





Let's now quantitatively analyze the LC circuit using conservation of energy. Recall for a capacitor energy is stored in the form of the electric field and for an inductor it is stored in the form of a magnetic field, and as we have seen above the strength of these fields oscillate over time. At any time, however, the total energy remains constant.

$$U_T = U_L + U_C$$
$$U_T = \frac{1}{2}Li^2(t) + \frac{1}{2C}q^2(t)$$

Differentiating this equation with respect to time we have.

$$\frac{dU_T}{dt} = \frac{1}{2}L\frac{d}{dt}i^2(t) + \frac{1}{2C}\frac{d}{dt}q^2(t)$$
$$0 = \frac{1}{\cancel{2}}L \cdot \cancel{2}i(t)\frac{di(t)}{dt} + \frac{1}{\cancel{2}C} \cdot \cancel{2}q(t)\frac{dq(t)}{dt}$$
$$0 = Li(t)\frac{di(t)}{dt} + \frac{1}{C}q(t)\frac{dq(t)}{dt}$$

Where, we used the fact that the total energy does not change with time, $\frac{dU_T}{dt} = 0$.

We now use the following substitutions so that we have an equation in terms of charge only.

$$i(t) = \frac{dq(t)}{dt} \qquad \frac{di(t)}{dt} = \frac{dq^2(t)}{dt^2}$$

$$0 = L\frac{dq(t)}{dt}\frac{dq^2(t)}{dt^2} + \frac{1}{C}q(t)\frac{dq(t)}{dt}$$
$$0 = \cancel{\frac{dq(t)}{dt}}\left(L\frac{dq^2(t)}{dt^2} + \frac{1}{C}q(t)\right)$$
$$\frac{dq^2(t)}{dt^2} = -\frac{1}{LC}q(t)$$

The final expression is a second order differential equation in the same form we saw when we solved the spring and mass system in Newtonian Mechanics. When solving the spring mass system, we came to the following conclusion.

Any physical system that can be represented by a differential equation of the form. $$\frac{d^2x(t)}{dt^2} = -C[x(t)]$$ Will results in *simple harmonic motion* and the position function can be written as: $$x(t) = A\cos(\omega t + \varphi)$$ Where the radial frequency is given by: $$\omega = \sqrt{C}$$ The constants, A and φ, are found by using initial conditions. E.g. $x(0) = x_0,\ x'(0) = x'_0$.

Applying the above principle to our differential equation for the LC circuit we can write.

$$q(t) = A\cos(\omega t + \varphi)$$

Where, $\omega = \frac{1}{\sqrt{LC}}$.

The current is then

$$i(t) = \frac{dq(t)}{dt} = -\omega A\sin(\omega t + \varphi)$$

Going back to our original qualitative example let's use the initial condition that the capacitor has an initial charge of Q at $t = 0$.

$$q(0) = A\cos(\omega 0 + \varphi)$$
$$Q = A\cos(\varphi)$$

If we let $\varphi = 0$, then $A = Q$, so that in this case we can write the charge and current equations as follows:

$$q(t) = Q\cos(\omega t)$$
$$i(t) = -\omega Q\sin(\omega t)$$

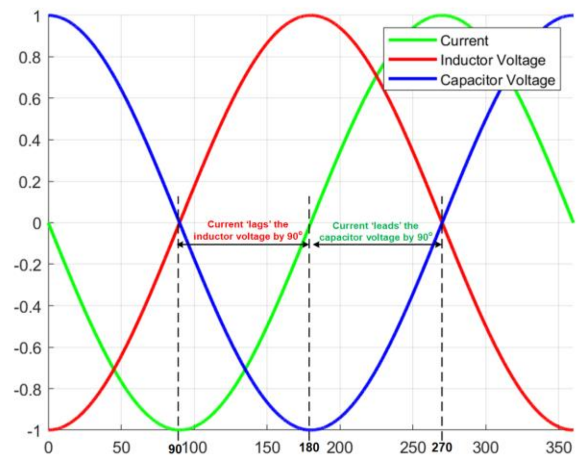
Let's now find the voltage across the inductor and capacitor. From our previous studies we know that the voltage across an inductor is proportional to the derivative of the current and the voltage across a capacitor is proportional to the integral of the current.

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(t) = -L\omega^2 Q\cos(\omega t)$$

$$V_C(t) = \frac{1}{C}\int_o^t i(\tau)d\tau + V_C(0)$$
$$V_C(t) = \frac{-\omega Q}{C}\int_o^t \sin(\omega\tau)\,d\tau + \frac{Q}{C}$$
$$V_C(t) = \frac{Q}{C}(\cos(\omega t) - 1) + \frac{Q}{C}$$
$$V_C(t) = \frac{Q}{C}\cos(\omega t)$$

We can now examine the relationship between these three quantities. Letting all amplitude quantities equal one, i.e. $\omega = Q = L = C = 1$, we can plot the following on the same figure.

$$i(t) = -\sin(\omega t) \qquad V_L(t) = -\cos(\omega t) \qquad V_C(t) = \cos(\omega t)$$



The figure tells us the following general current-voltage relationship for an inductor and a capacitor. Although not shown in the figure, the current-voltage relationship for a resistor is directly proportional to the current.

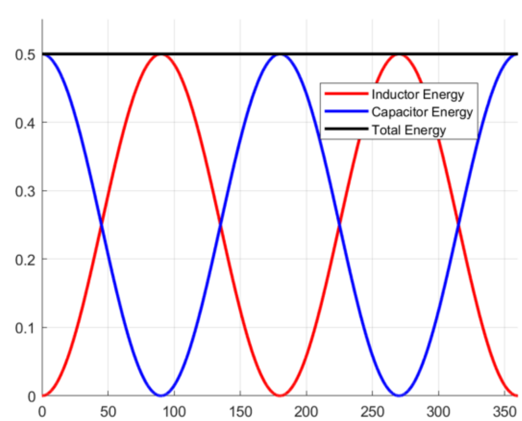
Inductor: The current *lags* the voltage by 90°.
Capacitor: The current *leads* the voltage by 90°.
Resistor: The current is in phase, (i.e. phase difference of 0°), with the voltage.

Finally, using the information from above we can look at how the energy oscillates in an LC circuit.

$$U_L = \frac{1}{2}Li^2(t) \qquad U_C = \frac{1}{2C}q^2(t)$$

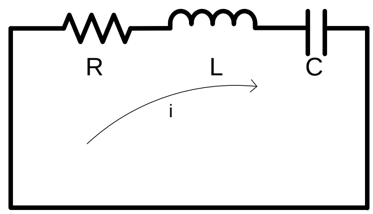
$$U_L = \frac{L\omega^2Q^2}{2}\sin^2(\omega t) \qquad U_C = \frac{Q}{2C}\cos^2(\omega t)$$

The figure below shows these energies along with the total energy, setting all amplitude value to one.



RLC Circuits

In the LC circuit from above no energy is lost because there is no resistive component. If we add a resistor, we could expect to get the same oscillatory behavior, but with each oscillation some energy would be dissipated through the resistor until all the energy is released. This general idea is indeed true, but let's analyze the series RLC circuit below to get a full quantitative understanding of the behavior.



We start by using Kirchhoff's voltage rule.

$$V_R + V_L + V_C = 0$$

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_o^t i(\tau)d\tau + V_C(0) = 0$$

$$\frac{d}{dt}\left[Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_o^t i(\tau)d\tau + V_C(0)\right] = \frac{d}{dt}[0]$$

$$R\frac{di(t)}{dt} + L\frac{di^2(t)}{dt} + \frac{1}{C}i(t) = 0$$

$$(1)\frac{di^2(t)}{dt} + \left(\frac{R}{L}\right)\frac{di(t)}{dt} + \left(\frac{1}{LC}\right)i(t) = 0$$

Where in the third line we differentiated both sides to remove the integral. The final equation is referred to as a second order linear constant coefficient differential equation, (LCCDE). This type of differential equation can be solved using various techniques. We present a general formulation in the appendix and utilize the results below.

The 2nd order LCCDE for a series RLC circuit is given as:

$$(1)\frac{di^2(t)}{dt} + \left(\frac{R}{L}\right)\frac{di(t)}{dt} + \left(\frac{1}{LC}\right)i(t) = 0$$

And from the appendix, the characteristic equation and its solutions can be written as follows:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad\qquad s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

For reasons that will become clear later we let $\alpha = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC}$ and rewrite the quadratic formula.

$$s_{1,2} = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega^2}}{2}$$

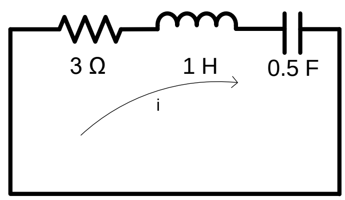
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$

From the appendix, the solution types can now be described as below.

Condition	***Roots***	***Characteristic***
$\alpha^2 > \omega^2$	$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$	Overdamped
$\alpha^2 < \omega^2$	$s_{1,2} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$	Underdamped
$\alpha^2 = \omega^2$	$s = -\alpha$	Critically Damped

Let's look at an example for each of the three cases above.

Overdamped



In this case $\alpha^2 = \frac{9}{4}$ and $\omega^2 = 2$. We see that $\alpha^2 > \omega^2$ and the roots are given as:

$$s_1 = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}$$

$$s_{1,2} = -1, -2$$

Therefore, the overdamped response is

$$i(t) = A_1 e^{-t} + A_2 e^{-2t}$$

We solve for A_1 and A_2 using current and voltage across the inductor at $t = 0$ as: $i(0) = 1,\ V_L(0) = 7$

$$i(0) = A_1 e^{-0} + A_2 e^{-2\cdot 0}$$

$$1 = A_1 + A_2$$

$$V_L(t) = L\frac{di(t)}{dt}$$

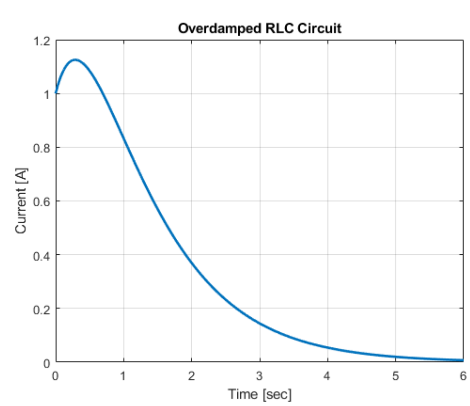
$$V_L(t) = L(-A_1 e^{-t} + -2A_2 e^{-2t})$$

$$V_L(0) = 1 \cdot (-A_1 e^{-0} - 2A_2 e^{-2\cdot 0})$$

$$7 = A_1 - 2A_2$$

Solving the above simultaneous equations, we find $A_1 = 3$ and $A_2 = -2$. Finally, the equation for the current, along with a plot for $t > 0$ is given below.

$$i(t) = 3e^{-t} - 2e^{-2t}$$



Underdamped

The roots of the characteristic equation for an underdamped system are.

$$s_{1,2} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$$

Substituting $\omega_d = \sqrt{\omega^2 - \alpha^2}$, which we call the *damped resonant frequency* we have

$$s_{1,2} = -\alpha \pm i\omega_d$$

The solution can then be written as

$$i(t) = A_1 e^{-\alpha t + i\omega_d t} + A_2 e^{-\alpha t - i\omega_d t}$$
$$i(t) = e^{-\alpha t}\left(A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right)$$

With the Euler identity, $e^{\pm i\omega_d t} = \cos(\omega_d t) \pm i\sin(\omega_d t)$, we can rewrite the solution as a sum of sinusoids.

$$i(t) = e^{-\alpha t}[A_1(\cos(\omega_d t) + i\sin(\omega_d t)) + A_2(\cos(\omega_d t) - i\sin(\omega_d t))]$$
$$i(t) = e^{-\alpha t}[(A_1 + A_2)\cos(\omega_d t) + (A_1 - A_2)i\sin(\omega_d t)]$$
$$i(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)]$$

Where, $B_1 = (A_1 + A_2)$ and $B_2 = i(A_1 - A_2)$, and since $i(t)$ must be real B_1 and B_2 must also be real.

Finally, we can once more rewrite $i(t)$ as a single cosine wave with a phase shift using the following trigonometric identity; with the addition of multiplying through by a new constant, A.

$$\mathrm{A}\cos(x - y) = \mathrm{A}\cos(x)\cos(y) + \mathrm{A}\sin(x)\sin(y)$$

Letting $B_1 = \mathrm{A}\cos(\theta)$ and $B_2 = \mathrm{A}\sin(\theta)$, we can rewrite $i(t)$ as

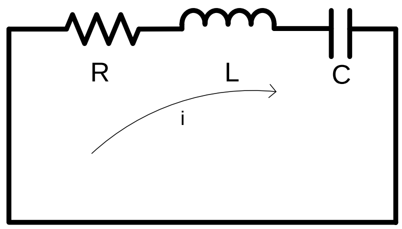
$$i(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)]$$
$$i(t) = e^{-\alpha t}[\mathrm{A}\cos(\theta)\cos(\omega_d t) + \mathrm{A}\sin(\theta)\sin(\omega_d t)]$$
$$i(t) = Ae^{-\alpha t}[\cos(\omega_d t - \theta)]$$

The two new constants, A and θ, can be related to B_1 and B_2 as follows.

$$\mathrm{A}^2\cos^2(\theta) + \mathrm{A}^2\sin^2(\theta) = {B_1}^2 + {B_2}^2$$
$$\mathrm{A}^2(\cos^2(\theta) + \sin^2(\theta)) = {B_1}^2 + {B_2}^2$$
$$\mathrm{A}^2(1) = {B_1}^2 + {B_2}^2$$
$$\mathrm{A} = \sqrt{{B_1}^2 + {B_2}^2}$$

$$\frac{\cancel{A}\sin(\theta)}{\cancel{A}\cos(\theta)} = \frac{B_2}{B_1}$$
$$\tan(\theta) = \frac{B_2}{B_1}$$
$$\theta = \tan^{-1}\left(\frac{B_2}{B_1}\right)$$

With the above results let's analyze the circuit with initial conditions; $i(0) = 1$, and $V_L(0) = 2\pi - 1$



$$R = 2\,\Omega$$

$$L = 1\,H$$

$$C = \frac{1}{4\pi^2 + 1}$$

From above we can find α^2 and ω^2 as follows:

$$\alpha^2 = \left(\frac{R}{2L}\right)^2$$
$$\alpha^2 = \left(\frac{2}{2 \cdot 1}\right)^2$$
$$\alpha^2 = 1$$

$$\omega^2 = \frac{1}{LC}$$
$$\omega^2 = \frac{1}{1 \cdot \left(\frac{1}{4\pi^2 + 1}\right)}$$
$$\omega^2 = 4\pi^2 + 1$$

Which confirms an underdamped system since $\alpha^2 < \omega^2$. The *damped resonant frequency* is then

$$\omega_d = \sqrt{\omega^2 - \alpha^2}$$
$$\omega_d = \sqrt{4\pi^2 + 1 - 1}$$
$$\omega_d = 2\pi$$

The final expression for current is found with the equation below and initial conditions.

$$i(t) = e^{-t}[B_1 \cos(2\pi t) + B_2 \sin(2\pi t)]$$
$$i(0) = e^{-0}[B_1 \cos(0) + \cancel{B_2 \sin(0)}]$$
$$1 = B_1$$

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(t) = 1 \cdot (e^{-t}[-2\pi B_1 \sin(2\pi t) + 2\pi B_2 \cos(2\pi t)] - e^{-t}[B_1 \cos(2\pi t) + B_2 \sin(2\pi t)])$$
$$V_L(0) = 1 \cdot (e^{-0}[-2\pi B_1 \sin(0) + 2\pi B_2 \cos(0)] - e^{-0}[B_1 \cos(0) + B_2 \sin(0)])$$
$$2\pi - 1 = 1 \cdot (2\pi B_2 - 1)$$
$$2\pi = 2\pi B_2$$
$$B_2 = 1$$

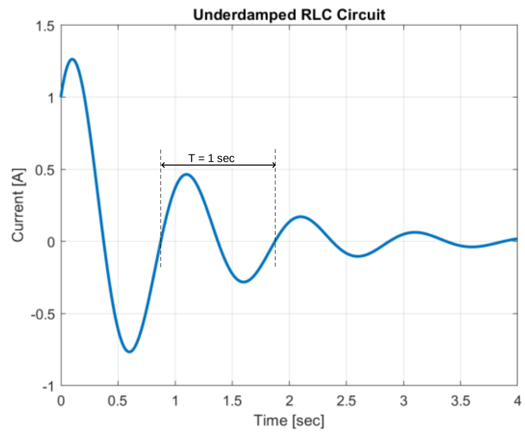
We can now use the results from above to find A and θ to write the current as one cosine function.

$$\mathrm{A} = \sqrt{{B_1}^2 + {B_2}^2}$$
$$\mathrm{A} = \sqrt{1^2 + 1^2}$$
$$\mathrm{A} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{B_2}{B_1}\right)$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$
$$\theta = 45^\circ$$

Finally, the equation for the current, along with a plot for $t > 0$ is given below. The figure shows the time period, $T = 1$, which is easily derived since $\omega_d \stackrel{\text{def}}{=} \frac{2\pi}{T}$.

$$i(t) = \sqrt{2}e^{-t}[\cos(2\pi t - 45^{\circ})]$$



Critically Damped

From the table above, we know for a critically damped system there is only one solution to the characteristic equation.

$$s_1 = -\alpha$$

The current can then be written as

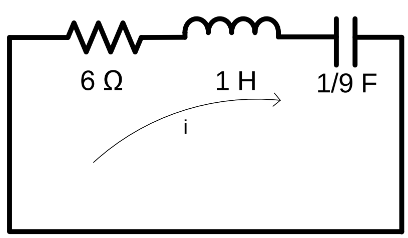
$$i(t) = A_1e^{-\alpha t} + A_2e^{-\alpha t}$$
$$i(t) = A_3e^{-\alpha t}$$

Where, $A_3 = A_1 + A_2$

This leaves us with one unknown constant, but we still have two initial conditions which demands a second solution. As the proof is beyond our scope here, we will simply state the general solution to a critically damped system as shown below.

$$i(t) = A_1te^{-\alpha t} + A_2e^{-\alpha t}$$

With this knowledge we analyze the circuit below with initial conditions given as follows: $i(0) = 1$, and $V_L(0) = 2$



From above we can find α^2 and ω^2 as follows:

$$\alpha^2 = \left(\frac{R}{2L}\right)^2$$
$$\alpha^2 = \left(\frac{6}{2 \cdot 1}\right)^2$$
$$\alpha^2 = 9$$

$$\omega^2 = \frac{1}{LC}$$
$$\omega^2 = \frac{1}{1 \cdot \left(\frac{1}{9}\right)}$$
$$\omega^2 = 9$$

Which confirms a critically damped system since $\alpha^2 = \omega^2$, therefore our general solution from above applies.

$$i(t) = A_1 te^{-\alpha t} + A_2 e^{-\alpha t}$$
$$i(t) = A_1 te^{-3t} + A_2 e^{-3t}$$

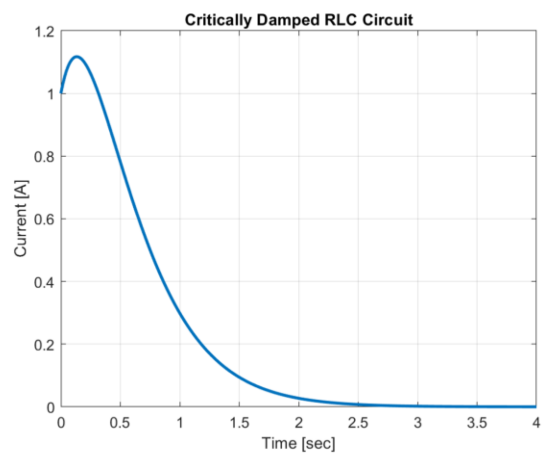
As usual we find the constants, A_1 and A_2, using the given initial conditions.

$$i(t) = A_1 te^{-3t} + A_2 e^{-3t}$$
$$i(0) = A_1 \cdot 0e^{-0} + A_2 e^{-0}$$
$$1 = A_2$$

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(t) = 1 \cdot (-3A_1 te^{-3t} + A_1 e^{-3t} - 3A_2 e^{-3t})$$
$$V_L(0) = 1 \cdot (-3A_1 \cdot 0 \cdot e^{-0} + A_1 e^{-0} - 3A_2 e^{-0})$$
$$V_L(0) = A_1 - 3A_2$$
$$2 = A_1 - 3$$
$$5 = A_1$$

Finally, the equation for the current, along with a plot for $t > 0$ is given below. The figure shows the time period, $T = 1$, which is easily derived since $\omega_d \stackrel{\text{def}}{=} \frac{2\pi}{T}$.

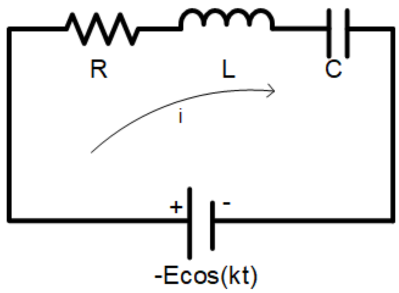
$$i(t) = 5te^{-3t} + e^{-3t}$$



Forced Response

The RLC circuits we looked at above did not have an applied voltage. The applied voltage is equivalent to the driving force, $f(t)$, explained in the appendix. In this section we focus on sinusoidal driving forces.

Let's take the underdamped circuit from above and add a sinusoidal voltage source.



$$R = 2\,\Omega$$

$$L = 1\,H$$

$$C = \frac{1}{4\pi^2 + 1}$$

We apply Kirchhoff's law as we did above but add in the source voltage

$$V_R + V_L + V_C = V_B$$

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_o^t i(\tau)d\tau + V_C(0) = -E\cos(kt)$$

$$\frac{d}{dt}\left[Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_o^t i(\tau)d\tau + V_C(0)\right] = \frac{d}{dt}[-E\cos(kt)]$$

$$R\frac{di(t)}{dt} + L\frac{di^2(t)}{dt} + \frac{1}{C}i(t) = Ek\sin(kt)$$

$$\frac{di^2(t)}{dt} + \left(\frac{R}{L}\right)\frac{di(t)}{dt} + \left(\frac{1}{LC}\right)i(t) = \left(\frac{Ek}{L}\right)\sin(kt)$$

We use the same substitutions as we did before, (i.e. $\alpha = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC}$). And for convenience we can also let $D = \frac{Ek}{L}$ and write the final equation as follows:

$$\frac{di^2(t)}{dt} + 2\alpha\frac{di(t)}{dt} + \omega^2 i(t) = D\sin(kt)$$

We know that the full response of this circuit if given as:

$$i(t) = i_n(t) + i_f(t)$$

And as we learned from above the natural response can be written as

$$i_n(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)]$$

Where, $\alpha = 1$, and $\omega_d = \sqrt{\omega^2 - \alpha^2} = 2\pi$

From the appendix the forced response is

$$i_f(t) = A_1 \cos(kt) + A_2 \sin(kt)$$

To solve for the constants A_1 and A_2 we need to substitute the forced response into the differential equation. Let's find the first and second derivatives first.

$$\frac{di_f(t)}{dt} = -A_1 k \sin(kt) + A_2 k \cos(kt)$$
$$\frac{di_f{}^2(t)}{dt} = -A_1 k^2 \cos(kt) - A_2 k^2 \sin(kt)$$

Substituting these into our differential equation we have

$$\frac{di^2(t)}{dt} + 2\alpha \frac{di(t)}{dt} + \omega^2 i(t) = D \sin(kt)$$
$$-A_1 k^2 \cos(kt) - A_2 k^2 \sin(kt) - 2\alpha A_1 k \sin(kt) + 2\alpha A_2 k \cos(kt) + \omega^2 A_1 \cos(kt) + \omega^2 A_2 \sin(kt)$$
$$= D \sin(kt)$$

We can solve for A_1 and A_2 by equating the coefficients of the sine and cosine terms.

Solving for A_1 using the cosine terms we have

$$-A_1 k^2 + 2\alpha A_2 k + \omega^2 A_1 = 0$$
$$A_1 = \frac{A_2(2\alpha k)}{(k^2 - \omega^2)}$$

Next, using the sine terms and substituting for A_1 we can find A_2 as follows

$$-A_2 k^2 - 2\alpha A_1 k + \omega^2 A_2 = D$$
$$-A_2 k^2 - \frac{A_2(2\alpha k)^2}{(k^2 - \omega^2)} + \omega^2 A_2 = D$$
$$A_2 = \frac{D}{1 - k^2 - \frac{(2\alpha k)^2}{(k^2 - \omega^2)}}$$
$$A_2 = \frac{D(k^2 - \omega^2)}{2k^2\omega^2 - \omega^4 - k^4 - (2\alpha k)^2}$$
$$A_2 = -\frac{D(k^2 - \omega^2)}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$

And A_1 is found as

$$A_1 = \frac{(2\alpha k)}{\cancel{(k^2 - \omega^2)}} \cdot \frac{-D\cancel{(k^2 - \omega^2)}}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$
$$A_1 = -\frac{D2\alpha k}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$

The forced response is then

$$i_f(t) = \left(-\frac{1}{(\omega^2 - k^2)^2 + (2\alpha k)^2}\right) D2\alpha k \cos(kt) + D(k^2 - \omega^2) \sin(kt)$$

The natural response constants, B_1 and B_2, can now be found using the initial conditions.

$$i(t) = i_n(t) + i_f(t)$$
$$i(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)] + A_1\cos(kt) + A_2\sin(kt)$$
$$i(0) = e^{-0}[B_1\cos(0) + \cancel{B_2\sin(0)}] + A_1\cos(0) + \cancel{A_2\sin(0)}$$
$$i(0) = B_1 + A_1$$
$$B_1 = i(0) - A_1$$

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(t) = 1(e^{-\alpha t}[-\omega_d B_1\sin(\omega_d t) + \omega_d B_2\cos(\omega_d t)] - \alpha e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)]) - A_1 \mathrm{k}\sin(kt) + A_2 k\cos(kt)$$
$$V_L(0) = (e^{-0}[\cancel{-\omega_d B_1\sin(0)} + \omega_d B_2\cos(0)] - \alpha e^{-0}[B_1\cos(0) + \cancel{B_2\sin(0)}]) - \cancel{A_1\mathrm{k}\sin(0)} + A_2 k\cos(0)$$
$$V_L(0) = \omega_d B_2 - \alpha B_1 + A_2 k$$
$$B_2 = \frac{V_L(0) + \alpha B_1 - A_2 k}{\omega_d}$$
$$B_2 = \frac{V_L(0) + \alpha(i(0) - A_1) - A_2 k}{\omega_d}$$

As you can see the algebra becomes quite tedious. In the next section we introduce an alternate method to find the forced solution. For now, the complete solution for $t \geq 0$ is written as

$$i(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)] + A_1\cos(kt) + A_2\sin(kt)$$

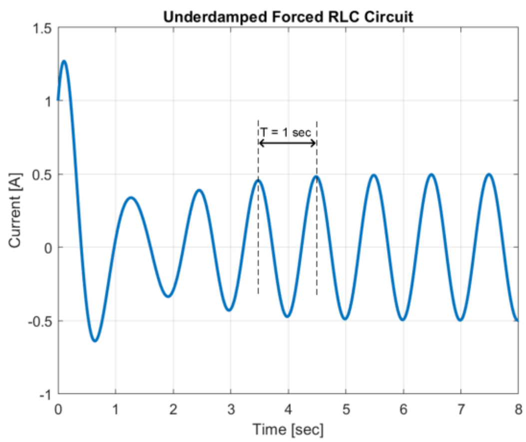
Where,

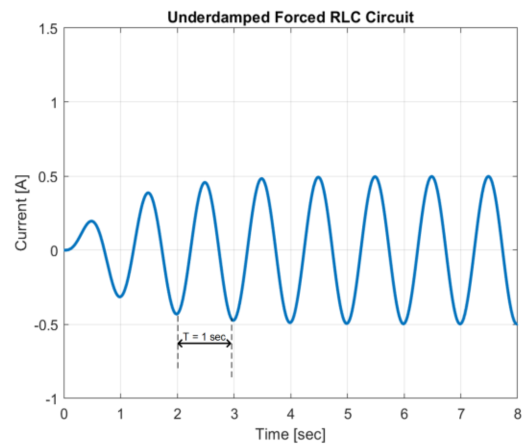
$$A_1 = -\frac{D2\alpha k}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$
$$A_2 = -\frac{D(k^2 - \omega^2)}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$
$$B_1 = i(0) - A_1$$
$$B_2 = \frac{V_L(0) + \alpha(i(0) - A_1) - A_2 k}{\omega_d}$$

To gain more insight, the figures below show the complete response using two different initial conditions. The figure on the left uses the same initial conditions from the original circuit without a source voltage. The figure on the right uses the initial conditions of $i(0) = V_L(0) = 0$. In both cases we use $k = 2\pi$. Examining the figures, we notice the following:

- Left Figure: $(i(0) = 1, V_L(0) = 2\pi - 1)$
    - The behavior from natural response is visible in the first approximately two seconds until the exponential term decays to the point where the forced response dominates.
    - The behavior from the natural response matches the behavior of the figure we plotted above when there was no voltage source.
    - For $t > 3$ seconds the forced response is dominate and the output is a scaled version of the input with the same frequency of $k = 2\pi$.

- Right Figure: $(i(0) = V_L(0) = 0)$
    - The current starts at zero and exponentially grows until the forced response again dominates.
    - For $t > 3$ seconds the forced response is dominate and the output is a scaled version of the input with the same frequency of $k = 2\pi$.





When the natural response decays enough so that the forced response dominates, we call this the *steady state response.* In most cases the initial transient response is of little concern. With this in mind let's look again at the coefficients, A_1 and A_2, of the steady state response.

$$A_1 = -\frac{D2\alpha k}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$

$$A_2 = -\frac{D(k^2 - \omega^2)}{(\omega^2 - k^2)^2 + (2\alpha k)^2}$$

Recall, that we can also write the steady state response using a single cosine wave as follows:

$$i(t) = A[\cos(kt - \theta)]$$

Where,

$$\mathrm{A} = \sqrt{{A_1}^2 + {A_2}^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{A_2}{A_1}\right)$$

Examining the magnitude term, A, we have.

$$\mathrm{A} = \sqrt{\left(-\frac{D2\alpha k}{(\omega^2 - k^2)^2 + (2\alpha k)^2}\right)^2 + \left(-\frac{D(k^2 - \omega^2)}{(\omega^2 - k^2)^2 + (2\alpha k)^2}\right)^2}$$

Note that the magnitude of the steady state response is a function of the input frequency, k. A plot of the magnitude as a function of the input frequency is called the *frequency response* of the RLC circuit. Let's look at what happens when the input frequency, k, matches the resonant frequency, ω, i.e. $k = \omega$.

$$\mathrm{A} = \sqrt{\left(-\frac{D2\alpha k}{(0)^2 + (2\alpha k)^2}\right)^2 + \left(-\frac{D(0)}{(0)^2 + (2\alpha k)^2}\right)^2}$$

$$\mathrm{A} = \sqrt{\left(-\frac{D2\alpha k}{(2\alpha k)^2}\right)^2} = \frac{D2\alpha k}{(2\alpha k)^2}$$

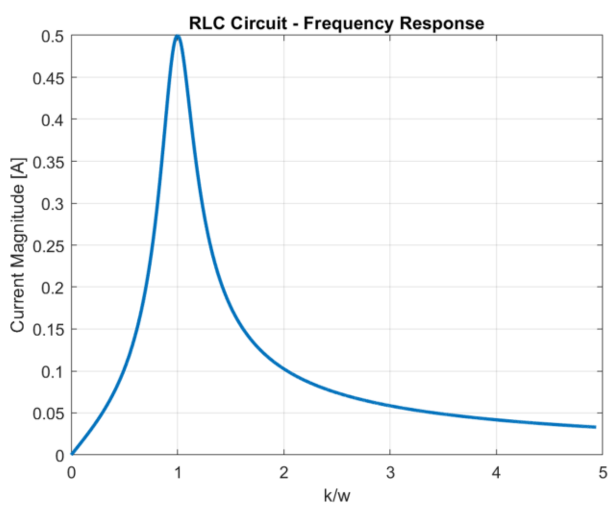
Substituting for D and α.

$$\mathrm{A} = \frac{\left(\frac{Ek}{L}\right)\cancel{2}\left(\frac{R}{\cancel{2}L}\right)k}{\left(\cancel{2}\frac{R}{\cancel{2}L}k\right)^2}$$

$$A = \left(\frac{E\cancel{R}\cancel{k^2}}{\cancel{L^2}}\right)\cdot\left(\frac{\cancel{L^2}}{R^{\cancel{2}}\cancel{k^2}}\right)$$

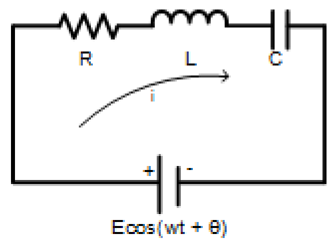
$$A = \frac{E}{R}$$

Which is the maximum output value of the current. This is the phenomenon of resonance and can be utilized to design circuits that acts as filters to pass desired frequencies and suppress undesired ones. The figure below is the magnitude plotted versus $k/_w$.



Phasor Method

We start with the same series RLC circuit from above with a general sinusoidal voltage source and assume the initial conditions are zero.



The conventional time domain differential equation describing this circuit as a result of applying Kirchhoff's law is

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_o^t i(\tau)d\tau = E\cos(\omega t + \theta)$$

We are interested in the steady state response, (i.e. the forced response) only, where the solution is assumed to be a sinusoid with the same frequency as the source voltage.

$$i(t) = A\cos(\omega t - \phi)$$

Referring to the appendix we first "transform" the source voltage and current from the conventional *time domain* representation to the so-called *phasor domain* representation.

$$\boldsymbol{V_B} = Ee^{i\theta} \qquad\qquad \boldsymbol{I} = Ae^{-i\phi}$$

Substituting we have

$$\boldsymbol{I}R + L\frac{d}{dt}\boldsymbol{I} + \frac{1}{C}\int_o^t \boldsymbol{I}d\tau = \boldsymbol{V_B}$$

Using results from the appendix we transform this differential equation into an algebraic equation.

$$\boldsymbol{I}(R) + \boldsymbol{I}(i\omega L) + \boldsymbol{I}\left(\frac{1}{i\omega C}\right) = \boldsymbol{V_B}$$

Note that the time dependence is not explicitly referenced in this equation, and if we instead treat ω as the independent variable we can say that we have transformed the equation from the *time domain* to the *frequency domain.* In many applications the frequency domain characteristics of a circuit are more important that the time domain. For example, frequency domain filters can be used to suppress unwanted high frequency noise from input signals. We can now solve using basic algebra.

$$\boldsymbol{I}\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] = \boldsymbol{V_B}$$

$$\boldsymbol{I} = \frac{\boldsymbol{V_B}}{\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right]}$$

$$\boldsymbol{I} = \frac{Ee^{i\theta}}{Be^{i\beta}}$$

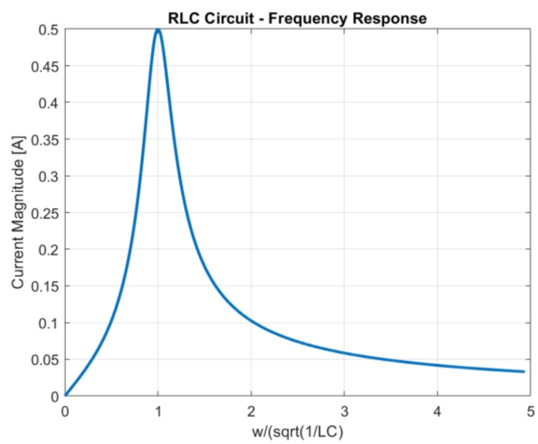
$$\mathrm{I} = I_M e^{i\phi}$$

Where,

$$I_M = \frac{E}{B} \qquad B = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\phi = \theta - \beta \qquad \beta = \tan^{-1}\left(\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}\right)$$

Setting $E = 1$ we can now plot the magnitude of I_M as a function of frequency. We again scale the x-axis by $\frac{1}{\sqrt{LC}}$, which is the resonant frequency of the RLC circuit. The plot is identical to the plot we displayed above using time domain analysis with differential equations instead of frequency domain analysis using algebra equations as we did here.



Impedance:

Let's take another look at the frequency dependent equation for our RLC circuit from above.

$$\boldsymbol{I}(R) + \boldsymbol{I}(i\omega L) + \boldsymbol{I}\left(\frac{1}{i\omega C}\right) = \boldsymbol{V_B}$$

Note the terms on the left-hand side represent the voltage drops across the resistor, inductor, and capacitor. The first term is Ohm's Law, which was previously used for DC, (direct current), circuit analysis in the time domain. As we have already shown, for circuits driven by AC, (alternating current), it is more convenient to use frequency domain analysis. With this in mind, we define the *impedance* of a circuit element as the ratio of the phasor voltage to the phasor current, which we denote by **Z**.

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$$

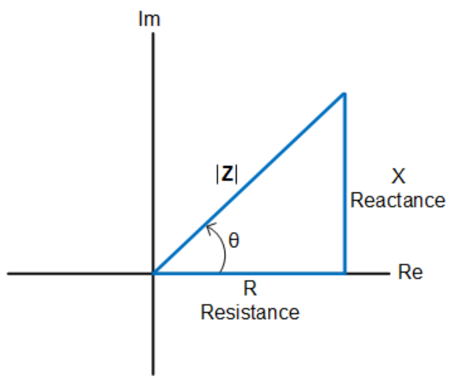
With this we can now extend Ohm's law using impedance in the frequency domain and apply it to each of the three elements above. The table below is a summary of these relationships.

Element Type	***Time Domain***	***Frequency Domain***	***Impedance (Z)***
Resistor	$v = Ri$	$\boldsymbol{V} = R\boldsymbol{I}$	$\boldsymbol{Z} = (R + i0)$
Inductor	$v = L\frac{di}{dt}$	$\boldsymbol{V} = i\omega L\boldsymbol{I}$	$\boldsymbol{Z} = (0 + i\omega L)$
Capacitor	$i = C\frac{dv}{dt}$	$\boldsymbol{V} = \frac{1}{i\omega C}\boldsymbol{I}$	$\boldsymbol{Z} = \left(0 - i\frac{1}{\omega C}\right)$

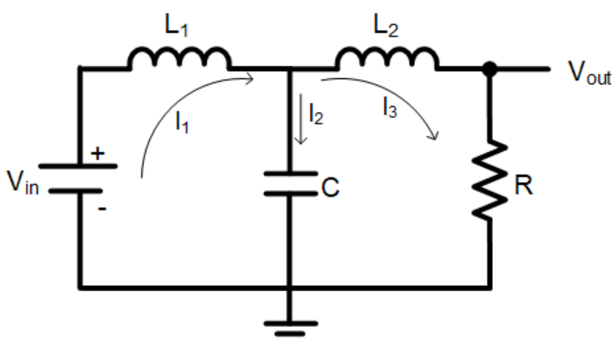
For AC circuit analysis the impedance is analogous to resistance for DC circuit analysis. The impedance is a complex number which can be written in various forms as shown below.

Polar Form	$\boldsymbol{Z} = \lvert\boldsymbol{Z}\rvert, \boldsymbol{\theta}$
Exponential Form	$\boldsymbol{Z} = \lvert\boldsymbol{Z}\rvert e^{i\theta}$
Rectangular Form	$\boldsymbol{Z} = R + iX$

We call the real part, $Re\{\boldsymbol{Z}\} = R$, the resistive part and the imaginary part, $Im\{\boldsymbol{Z}\} = X$, the reactive part. It can be graphically represented in the complex plane as shown below.



As mentioned, one important application of RLC circuits is for filtering. The circuit below is an example of a 3rd order Butterworth low pass filter, i.e. it suppresses high frequency signals while passing lower frequency signals. To illustrate how we can utilize what we learned from above let's find the steady state solution for the output voltage of the filter shown below.



$V_{in} = E\cos(\omega t)$

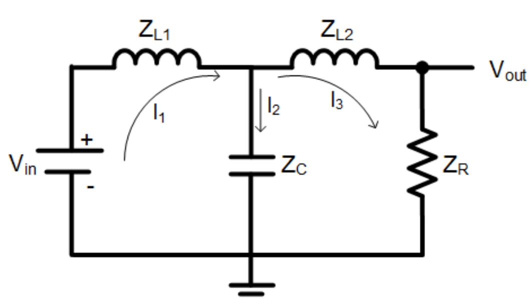
$L_1 = 3/2\,H$

$L_2 = 1/2\,H$

$C = 4/3\;F$

$R = 1\Omega$

The first step is to transform this filter from the time domain to the frequency domain so that we can write algebra equations instead of differential equations describing its behavior. We do this by redrawing the circuit with impedances as shown below.



Now we can perform conventional circuit analysis using Kirchhoff's laws as we have previously done with DC resistor circuits.

Kirchhoff's current rule for the junction at the top of the circuit is

$$I_1 = I_2 + I_3$$

And Kirchhoff's voltage law applied around the two loops in the circuit are as follows:

Left Loop	**Right Loop**
$V_{in} - I_1 Z_{L1} - I_2 Z_C = 0$	$I_2 Z_C - I_3 Z_{L2} - I_3 Z_R = 0$

To solve for I_3 we can start with the left loop equation, use the current rule to substitute for I_1, and then solve for I_2.

$$V_{in} - I_2 Z_{L1} - I_3 Z_{L1} - I_2 Z_C = 0$$

$$I_2 = \frac{V_{in} - I_3 Z_{L1}}{(Z_{L1} + Z_C)}$$

Now we can substitute for I_2 in the right loop equation and solve for I_3.

$$\frac{V_{in} Z_C}{(Z_{L1} + Z_C)} - \frac{I_3 Z_{L1} Z_C}{(Z_{L1} + Z_C)} - \frac{I_3 Z_{L2}(Z_{L1} + Z_C)}{(Z_{L1} + Z_C)} - \frac{I_3 Z_R (Z_{L1} + Z_C)}{(Z_{L1} + Z_C)} = 0$$

$$I_3 \left(\frac{Z_{L1} Z_C + Z_{L2} Z_{L1} + Z_{L2} Z_C + Z_R Z_{L1} + Z_R Z_C}{\cancel{(Z_{L1} + Z_C)}} \right) = \frac{V_{in} Z_C}{\cancel{(Z_{L1} + Z_C)}}$$

$$\frac{V_{in} Z_C}{(Z_{L1} Z_C + Z_{L2} Z_{L1} + Z_{L2} Z_C + Z_R Z_{L1} + Z_R Z_C)} = I_3$$

Finally, the output voltage is

$$V_{out} = I_3 Z_R = \frac{V_{in} Z_C Z_R}{(Z_{L1} Z_C + Z_{L2} Z_{L1} + Z_{L2} Z_C + Z_R Z_{L1} + Z_R Z_C)}$$

The procedure was straightforward, but since the inductor and capacitor impedances are imaginary, final computations are still somewhat tedious. Generally, with a filter we would like to plot the magnitude of the output voltage versus frequency. This is most easily done with excel or some other software computing tool. The Butterworth filter specified above was designed to have a cutoff frequency at $\omega = 1$, which means that the magnitude of the output voltage should be $\sqrt{2}/2$ times the input voltage when the input frequency is $\omega = 1$. Let's substitute values to verify this.

$$V_{out} = V_{in} \left[\frac{\left(\frac{-i}{\omega C}\right) R}{\left((i\cancel{\omega} L_1)\left(\frac{-i}{\cancel{\omega} C}\right) + (i\omega L_2)(i\omega L_1) + (i\cancel{\omega} L_2)\left(\frac{-i}{\cancel{\omega} C}\right) + R(i\omega L_1) + R\left(\frac{-i}{\omega C}\right) \right)} \right]$$

$$V_{out} = E e^{i0} \left[\frac{\left(\frac{-iR}{\omega C}\right)}{\left(\left(\frac{L_1}{C}\right) - (\omega^2 L_2 L_1) + \left(\frac{L_2}{C}\right) + (i\omega R L_1) - \left(\frac{iR}{\omega C}\right) \right)} \right]$$

$$V_{out} = E\left[\frac{\left[0 - i\left(\frac{R}{\omega C}\right)\right]}{\left[\left(\frac{L_1}{C}\right) - (\omega^2 L_2 L_1) + \left(\frac{L_2}{C}\right)\right] + i\left[(\omega R L_1) - \left(\frac{R}{\omega C}\right)\right]}\right]$$

$$V_{out} = E\left[\frac{\left[0 - i\left(\frac{3}{4}\right)\right]}{\left[\left(\frac{9}{8}\right) - \left(\frac{6}{8}\right) + \left(\frac{3}{8}\right)\right] + i\left[\left(\frac{6}{4}\right) - \left(\frac{3}{4}\right)\right]}\right]$$

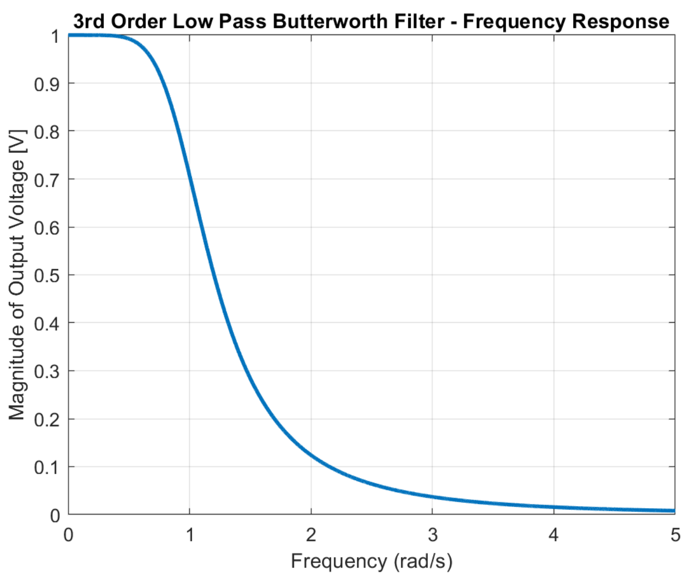
$$V_{out} = E\,\frac{\left[0 - i\left(\frac{3}{4}\right)\right]}{\left[\left(\frac{3}{4}\right)\right] + i\left[\left(\frac{3}{4}\right)\right]} = E\left(\frac{\left[-\frac{9}{16} - i\frac{9}{16}\right]}{\frac{18}{16}}\right)$$

The gain, G, of the filter at $\omega = 1$ is then given by

$$G = \frac{16}{18}\left(\sqrt{\left(\frac{9}{16}\right)^2 + \left(\frac{9}{16}\right)^2}\right) = \frac{\cancel{16}}{\cancel{18}\,(2)}\left(\sqrt{2}\right)\left(\frac{\cancel{9}\,(1)}{\cancel{16}}\right) = \frac{\sqrt{2}}{2}$$

Just as we expected!

Finally, below is a plot of the magnitude of the output voltage versus frequency computed using a software computing tool. As expected, the filter will pass low frequency signals and suppress high frequency ones.



Appendix:

1.) Method to Solve Second Order Liner Constant Coefficient Differential Equations

The general form of 2nd order linear constant coefficient differential equation (LCCDE) can be written as follows:

$$(a_2)\frac{dx^2(t)}{dt^2} + (a_1)\frac{dx(t)}{dt} + (a_0)x(t) = f(t)$$

Where, $f(t)$ is a called a forcing function, (e.g. a voltage source in the context of circuits). The solution to this equation can in general be written as the sum of a so-called *natural response* and a *forced response*.

$$x(t) = x_n(t) + x_f(t)$$

The natural response, which we will focus on first, satisfies the differential equation when $f(t) = 0$.

$$(a_2)\frac{dx^2(t)}{dt} + (a_1)\frac{dx(t)}{dt} + (a_0)x(t) = 0$$

We start by postulating a solution of the form: Ae^{st}. Differentiating this we have.

$$\frac{dx_n(t)}{dt} = sAe^{st} \qquad \frac{d{x_n}^2(t)}{dt^2} = s^2Ae^{st}$$

Substituting our postulated solution into the original equation we have the following.

$$a_2s^2Ae^{st} + a_1sAe^{st} + a_0Ae^{st} = 0$$
$$Ae^{st}(a_2s^2 + a_1s + a_0) = 0$$

Ignoring the trivial solution, $Ae^{st} = 0$, we are left with the so-called *characteristic equation* for a 2nd order LCCDE.

$$a_2s^2 + a_1s + a_0 = 0$$

Which can be solved for s using the quadratic formula.

$$s_{1,2} = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_2a_0}}{2a_2}$$

Therefore, the natural response is given as

$$x_n(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Where, A_1 and A_2 can be determined from initial conditions.

Let's look closer at the types of solutions that are possible. The characteristic equation is called such because the solutions to this equation, $s_{1,2}$, will determine the character of the natural response. Recall, there are three types of roots for a quadratic equation that are solely determined by discriminate of the quadratic equation. The discriminate, D, is the term inside the square root.

$$D = {a_1}^2 - 4a_2a_0$$

The three types of solutions are listed below.

Discriminant	***Roots***	***Characteristic***
$D > 0$	2 Distinct Real Roots	Overdamped
$D < 0$	2 Roots that are Complex Conjugates	Underdamped
$D = 0$	2 Repeated real roots	Critically Damped

Forced Response

The forced response satisfies the differential equation when $f(t) \neq 0$.

$$(a_2)\frac{dx^2(t)}{dt} + (a_1)\frac{dx(t)}{dt} + (a_0)x(t) = f(t)$$

For the natural response we postulated a solution of the form Ae^{st}. This made sense because when $f(t) = 0$ we can write the differential equation as follows:

$$(a_2)\frac{dx^2(t)}{dt} + (a_1)\frac{dx(t)}{dt} = (-a_0)x(t)$$

Written in this way we see that to satisfy this equation a function is needed such that when we take the derivative, a scaled version of the original function is the result. The exponential function fits this criterion.

Thinking about the forced response problem in the same way we can see, for example, that if the forcing function is a constant, the solution must also be a constant. This can be seen by assuming a solution other than a constant, e.g. $x(t) = A_1t + A_2$, when the forcing function is $f(t) = E$.

$$(a_2)\frac{dx^2(t)}{dt} + (a_1)\frac{dx(t)}{dt} + (a_0)x(t) = f(t)$$

$$(a_2) \cdot 0 + (a_1)A_1 + (a_0)(A_1t + A_2) = E$$

$$a_1A_1 + a_0(A_1t + A_2) = E$$

The equality is satisfied only when $A_1 = 0$ and $A_2 = E/a_0$, which results in a constant solution as we expected.

$$x(t) = \frac{E}{a_0}$$

Extending on this concept, the table below shows a list of some forcing functions and assumed solutions.

Forcing Function	***Assumed Solution***
E	A
Et	$A_1 t + A_2$
Et^2	$A_1 t^2 + A_2 t + A_3$
Ee^{-st}	Ae^{-st}
$E \sin(\omega t)$	$A_1 \cos(\omega t) + A_2 \sin(\omega t)$

The constants in the assumed solution can be found using the technique from above, that is, substitute the assumed solution into the differential equation and equate the common terms. Finally, the complete solution is given as the addition of the natural and forced response.

$$x(t) = x_n(t) + x_f(t)$$

2.) Phasor Representation of Sinusoids

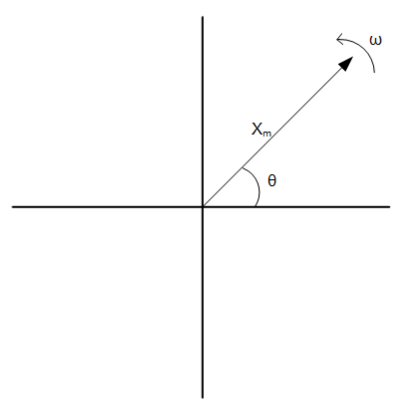
Phasors are used to represent sinusoids of the same frequency but with different amplitudes and phases. Regarding RLC circuits, we use phasors to more easily solve for the steady state response. The major advantage of using phasors is that we can "*transform*" the differential equations describing the circuits into algebraic equations. We start with the time domain representation of a sinusoid.

$$x(t) = X_m \cos(\omega t + \theta)$$

Using Euler's Identity, $e^{i(\omega t+\theta)} = \cos(\omega t + \theta) + i \sin(\omega t + \theta)$, we can also express the sinusoid as the real part of a complex exponential function.

$$x(t) = Re\{X_m e^{i\theta} e^{i\omega t}\}$$

The complex exponential function can be though of as a vector of length X_m, with an initial angle of θ rotating counterclockwise in 2D plane with a frequency of ω shown below.



To simplify the notation even more, we can drop the Re notation as well as the time dependent component, $e^{i\omega t}$, and express $x(t)$ in *phasor notation* as:

$$\boldsymbol{X} = X_m e^{i\theta}$$

And keep note of the value of the frequency, ω, for later use.

As mentioned, phasors allow for easier computations when dealing with LCCDE. With this in mind, we demonstrate differentiation and integration for phasors below.

Differentiation:

$$y(t) = \frac{d}{dt}\left[Re\{X_m e^{i\theta} e^{i\omega t}\}\right]$$
$$y(t) = Re\left\{X_m e^{i\theta} \frac{d}{dt}\left(e^{i\omega t}\right)\right\}$$
$$y(t) = Re\{i\omega X_m e^{i\theta} e^{i\omega t}\}$$

Using phasor notation from above we again drop the Re notation and $e^{i\omega t}$.

$$\boldsymbol{Y} = i\omega\left(X_m e^{i\theta}\right)$$
$$\boldsymbol{Y} = i\omega \boldsymbol{X}$$

Using $i = e^{i\frac{\pi}{2}}$, we can see that the derivative of a sinusoid produces another sinusoid multiplied by a constant, ω, and shifted by +90 degrees. Of course, this fact can also be shown using conventional time domain differentiation of a sinusoid.

Integration:

$$y(t) = \int Re\{X_m e^{i\theta} e^{i\omega t}\}\, dt$$
$$y(t) = Re\left\{X_m e^{i\theta} \int \left(e^{i\omega t}\right) dt\right\}$$
$$y(t) = Re\left\{\frac{1}{i\omega} X_m e^{i\theta} e^{i\omega t}\right\}$$

Just as we did above we drop the Re notation and $e^{i\omega t}$ for phasor notation.

$$\boldsymbol{Y} = \frac{1}{i\omega}\left(X_m e^{i\theta}\right)$$
$$\boldsymbol{Y} = \frac{1}{i\omega}\boldsymbol{X}$$

Which again shows the that integral of a sinusoid produces another sinusoid multiplied by a constant, $\frac{1}{\omega}$, and shifted by -90 degrees

Therefore, the derivative/integral of a phasor becomes multiplication/division by $i\omega$.

Derivative of a Phasor	***Integral of a Phasor***
$\frac{d}{dt}(\boldsymbol{X}) = i\omega \boldsymbol{X}$	$\int \boldsymbol{X}\, dt = \frac{1}{i\omega}\boldsymbol{X}$

By: ferrantetutoring