Inductors and Inductance Introduction

Mutual-Inductance

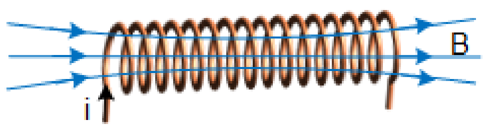
Faraday's law tells us that an EMF can be induced around a closed path in a conductor if the magnetic flux through the surface enclosed by the path changes.

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

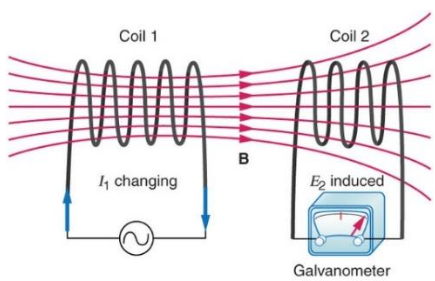
We also previously learned that a magnetic field is established when a current is made to flow in a conductor. For example, the solenoid, as you will recall, has a magnetic field directed as shown below with a magnitude given as:

$$B = \mu_0 ni, \text{ with } n = N/l$$

Where, l is the length of the coil.



Using these two observations, the configuration below can be used to establish an EMF in a second coil by applying a time varying current in a first coil.



The amount of magnetic flux created in the second coil per unit of current from the first is called the *mutual inductance* and is defined as follows:

$$M_{21} = \frac{\mathrm{N}_2\Phi_{B,1}}{i_1}$$

Where, M_{21} refers to the inductance of coil 2 with respect to coil 1.

Circling back to Faraday's law, the EMF induced in the second coil is as follows:

$$\varepsilon_2 = -N_2\frac{d}{dt}\left(\Phi_{B,1}\right)$$
$$\varepsilon_2 = -\cancel{N_2}\frac{d}{dt}\left(\frac{M_{21}i_1}{\cancel{\mathrm{N}_2}}\right)$$
$$\varepsilon_2 = -M_{21}\frac{di_1}{dt}$$

Although we will not prove it here it turns out that $M_{21} = M_{12}$, and therefore depending on which coil is providing the time varying current we have.

Coil 1 induces an EMF in coil 2	**Coil 2 induces an EMF in coil 1**
$\varepsilon_2 = -M\dfrac{di_1}{dt}$	$\varepsilon_1 = -M\dfrac{di_2}{dt}$

The concept of mutual inductance has many practical applications, one of which is that of an electrical transformer. As an example, electrical power delivery relies on transformers to deliver relatively low voltage energy to homes from much higher voltage power lines.

Self-Inductance

The concept of inductance can also be applied to a single isolated coil. When a time varying current is produced in a coil the magnetic flux through that coil is changing, which in turn will induce a current, (to oppose the change in flux - "Lenz's law"), in the coil itself. As an analogy to the mutual inductance we can define *self-inductance* as below.

$$L = \frac{N\Phi_B}{i}$$

Where, N is the number of turns in the coil.

Solving for the EMF due to self-inductance just as we did with mutual inductance we have:

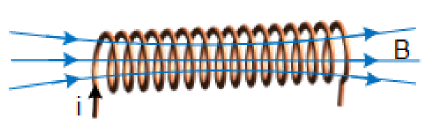
$$\varepsilon_L = -L\frac{di}{dt}$$

In an earlier section we introduced the capacitor, which is a device commonly used in electronic circuits that stores energy in the form of an electric field. An *inductor* is a device that is also commonly used in electronic circuits that stores energy in the form of a magnetic field. One of the simplest types of inductors is a solenoid. The amount of charge stored in a capacitor per unit voltage applied was referred to as the *capacitance,* $C = Q/V$. Similarly, as defined above, the amount of magnetic flux produced per unit of current is called the *inductance,* $L = \frac{N\Phi_B}{i}$.

Note that we were able to write the capacitance of a parallel plate capacitor in terms of its geometry only. As it turns out, we can do the same for inductors. The solenoid below, as you will recall, has a magnetic field directed as shown with a magnitude given as:

$$B = \mu_0 ni, \text{ with } n = N/l$$

Where, l is the length of the coil.



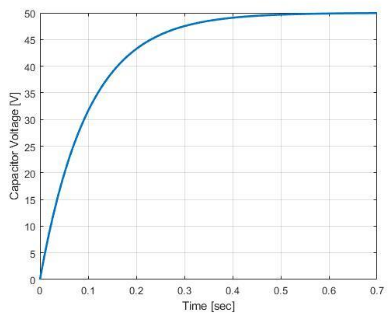
The magnetic flux through the opening surface of the coil is $\Phi_B = BA = \mu_0 niA$. Therefore, the inductance of the coil is as follows:

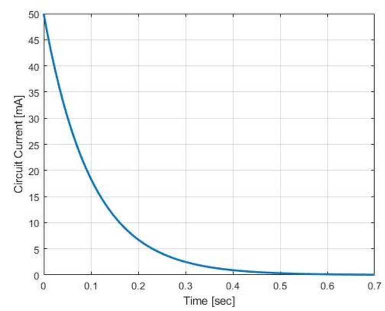
$$L = \frac{N\Phi_B}{i}$$
$$L = \frac{nl(\mu_0 n\cancel{i}A)}{\cancel{i}}$$
$$L = \mu_0 n^2 Al$$

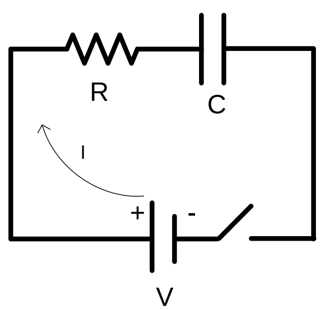
Inductors in Circuits

When studying RC circuits, we noticed there exists a transient time immediately after we apply a voltage until the capacitor is fully charged. For the simple RC circuit shown below we found the following behavior.

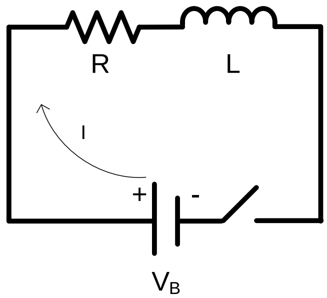
Charging a Capacitor in an RC Circuit	
Voltage across Capacitor	*Current in Circuit*
$v(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$	$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}$







We can see that the voltage across the capacitor increases as more charge builds on the capacitor plates creating a stronger electric field. Conversely the current decays until it finally stops flowing and the supply voltage drops entirely across the capacitor. Now let’s analyze what happens when we replace the capacitor with an inductor.



Just as we did with the RC circuit, we apply Kirchhoff's voltage rule around the loop.

$$V_B - i(t)R - L\frac{di(t)}{dt} = 0$$

The result is a fist order differential equation, just as we had with an RC circuit, which can be similarly solved using the separation of variables technique.

$$V_B - iR = L\frac{di}{dt}$$

$$\frac{1}{V_B - iR} = \frac{1}{L}\frac{dt}{di}$$

$$\left(\frac{1}{V_B - iR}\right) di = \frac{1}{L}\, dt$$

$$\int_0^i \left(\frac{1}{V_B - iR}\right) di = \int_0^t \frac{1}{L}\, dt$$

$$-\frac{1}{R}\ln(V_B - iR)|_0^i = \frac{1}{L}t$$

$$-\frac{1}{R}(\ln(V_B - iR) - \ln(V_B)) = \frac{1}{L}t$$

$$\ln\left(\frac{V_B - iR}{V_B}\right) = -\frac{R}{L}t$$

$$1 - \frac{iR}{V_B} = e^{-\frac{R}{L}t}$$

$$i(t) = \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

In this case we find that the *current grows* over time until it reaches a steady state value of V_B/R. At this point the magnetic flux stops changing and the inductor acts as a simple wire, (i.e. a short circuit), so that the current is limited by the resistor only.

The voltage across the inductor is then

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(t) = L\frac{d}{dt}\left(\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right)$$
$$V_L(t) = \frac{\cancel{L}V}{\cancel{R}}\left(0 + \frac{\cancel{R}}{\cancel{L}}e^{-\frac{R}{L}t}\right)$$
$$V_L(t) = Ve^{-\frac{R}{L}t}$$

Which shows that the voltage across the inductor exponentially decays until the entire supply voltage drops across the resistor only.

Finally, we can find the energy stored in the magnetic field of the inductor. Recall the energy stored in a capacitor when a voltage, V, is applied across was found to be

$$U_C = \frac{1}{2}CV^2$$

The energy stored in an inductor is completely analogous and can be found in a similar fashion. In this case let's take our differential equation from above and multiply through by the current.

$$Vi = i^2R + Li\frac{di}{dt}$$

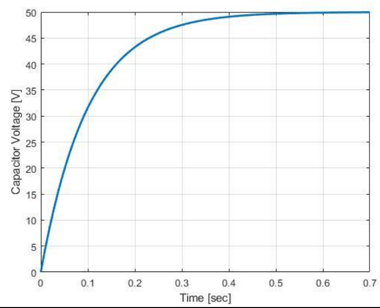
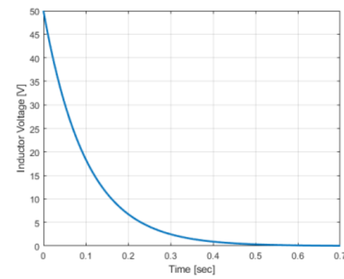
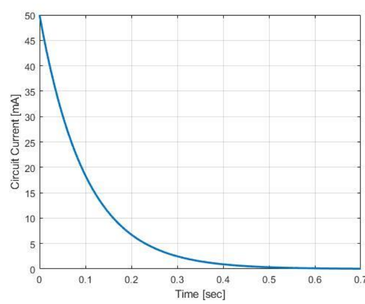
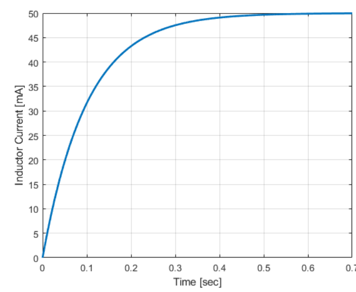
Since the power is given as $P = Vi$, we can interpret this relationship as follows:

The battery is supplying energy to the circuit at a rate of Vi, which is being dissipated by the resistor at a rate of i^2R and being stored, (in the form of a magnetic field), by the inductor at a rate of $Li\frac{di}{dt}$. Therefore, we have the following.

$$\frac{dU_L}{dt} = P_L$$
$$\frac{dU_L}{\cancel{dt}} = Li\frac{di}{\cancel{dt}}$$
$$\int_0^{U_L} dU_L = \int_0^i Li\,di$$
$$U_L = \frac{1}{2}Li^2$$

As you can see the energy stored in an inductor is completely analogous to the energy stored in a capacitor.

As you may have noticed the transient behavior of an RL circuit is also analogous to that of the RC circuit we previously studied. Below is a side by side comparison of the two types of components.

Property	***RC Circuit***	***RL Circuit***
Time Constant	$\tau = RC$	$\tau = \frac{L}{R}$
Component Voltage Behavior		
Component Current Behavior		
Charging Equations	$i_C(t) = \frac{V}{R} e^{-\frac{1}{RC}t}$ $V_C(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$	$i_L(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ $V_L(t) = Ve^{-\frac{R}{L}t}$
General Voltage-Current Relationship	$i_C(t) = C\frac{dV_C(t)}{dt}$ $V_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C(\tau)d\tau + V_C(t_0)$	$V_L(t) = L\frac{di(t)}{dt}$ $i_L(t) = \frac{1}{L}\int_{t_0}^{t} V_L(\tau)d\tau + i_L(t_0)$
Energy Stored	$U_C(t) = \frac{1}{2}CV_C(t)^2$	$U_L(t) = \frac{1}{2}Li(t)^2$

Summary for Inductance, Inductors, and RL Circuits

Mutual Inductance

The amount of magnetic flux created in a second coil per unit of current from a first coil is called the *mutual inductance* and is defined as follows:

$$M_{21} = \frac{\mathrm{N}_2 \Phi_{B,1}}{i_1}$$

The EMF induced in a second coil from a first coil is given as follows:

$$\varepsilon_{1/2} = -M \frac{di_{2/1}}{dt}$$

Where, $M_{21} = M_{12} = M$

Self-Inductance

When a time varying current is produced in a coil the magnetic flux through that coil is changing, which in turn will induce a current, (to oppose the change in flux - "Lenz's law"), in the coil itself. As an analogy to the mutual inductance we can define *self-inductance* as below.

$$L = \frac{\mathrm{N}\Phi_B}{i}$$

Where, N is the number of turns in the coil.

The EMF due to self-inductance is given as follows:

$$\varepsilon_L = -L \frac{di}{dt}$$

Inductor Energy

Energy stored in an inductor, in the form of a magnetic field, is given by the following:

$$U_L(t) = \frac{1}{2} L i(t)^2$$

RL Circuits

When a voltage source, V_B, is applied to a resistor and inductor in series the current and voltage vary according to the following equations.

$$V_L(t) = V_B e^{-\frac{R}{L}t} \qquad i_L(t) = \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

RL Time Constant

$$\tau = \frac{L}{R}$$

Examples:

Question 1:

Coil 1 has $L_1 = 25\ mH$ and $N_1 = 100$ turns. Coil 2 has $L_1 = 40\ mH$ and $N_2 = 200$ turns. The coils are fixed in place; their mutual inductance is $M = 3\ mH$. A 6 mA current in coild 1 is changing at a rate of $4\ A/s$.

a.) What is the magnetic flux that links coil 2, $\Phi_{2,1}$.
b.) What is the magnitude of the EMF induced in coil 2?
c.) What is the magnitude of the self-induced EMF in coil 1?

Solution 1:

Part a.)

The magnetic flux through coil 2 from coil 1 divided by the current from coil 1 is defined as the mutual inductance, which we know to be $3\ mH$.

$$M = \frac{\mathrm{N}_2 \Phi_{2,1}}{i_1}$$
$$\Phi_{2,1} = \frac{M i_1}{\mathrm{N}_2}$$
$$\Phi_{2,1} = \frac{(3E^{-3}) \cdot (6E^{-3})}{200}$$
$$\Phi_{2,1} = 9E^{-8}\ T \cdot m^2$$

Part b.)

The EMF induced in coil 2 is proportional to the rate of change of the current in coil 1, and the mutual inductance is the proportionality constant.

$$\varepsilon_2 = M \frac{di_1}{dt}$$
$$\varepsilon_2 = (3E^{-3}) \cdot 4$$
$$\varepsilon_2 = 0.012\ V$$

Part c.)

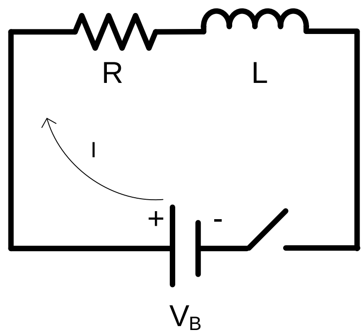
The self-induced EMF in coil 1 is also proportional to the rate of change of the current in coil 1, but with L as the proportionality constant.

$$\varepsilon_L = L \frac{di_1}{dt}$$
$$\varepsilon_L = (25E^{-3}) \cdot 4$$
$$\varepsilon_L = 0.1\ V$$

Question 2:

In the circuit shown below, with $R = 3\Omega$ and $L = 4H$, the switch has been open for a long time. At time $t = 0$, the switch is closed. Answer the following questions.

a.) Find the time when the current through the inductor is 50% of its maximum value.
b.) Find the rate at which energy is being dissipated through the resistor and being stored in the inductor when the elapsed time is one time constant.
c.) Find the total energy stored in the inductor and the total energy dissipated from the resistor when the elapsed time is five time constants.



Solution 2:

Part a.)

The current through the inductor at any time $t > 0$ is given as.

$$i(t) = \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Where, $\frac{V_B}{R}$ is the maximum value as $t \to \infty$.

Therefore, we can solve for the time the current reaches $0.5\frac{V}{R}$ as follows:

$$0.5\frac{\cancel{V_B}}{\cancel{R}} = \frac{\cancel{V_B}}{\cancel{R}}\left(1 - e^{-\frac{R}{L}t}\right)$$
$$e^{-\frac{R}{L}t} = (1 - 0.5)$$
$$-\frac{R}{L}t = \ln(0.5)$$
$$t = -\frac{L}{R}\ln(0.5)$$
$$t = -\frac{3}{4}\ln(0.5)$$
$$t = 0.52\ s$$

Part b.)

Recall, when deriving the energy stored in an inductor, we took our differential equation and multiplied through by the current. When we did this, we ended up with a power relationship as shown below.

$$V_B i(t) = i(t)^2 R + L i(t) \frac{di(t)}{dt}$$
$$P_B(t) = P_R(t) + P_L(t)$$

Using the equation for the current we can find new expressions for $P_R(t)$ and $P_L(t)$.

$$P_R(t) = i(t)^2 R$$
$$P_R(t) = \left[\frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right]^2 R$$
$$P_R(t) = \frac{{V_B}^2}{R}\left(1 - e^{-\frac{R}{L}t}\right)^2$$

$$P_L(t) = L i(t) \frac{di(t)}{dt}$$
$$P_L(t) = L \cdot \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)\left[\frac{V}{R}\left(0 + \frac{R}{L} e^{-\frac{R}{L}t}\right)\right]$$
$$P_L(t) = \cancel{L} \cdot \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)\left[\frac{V}{\cancel{R}}\left(\frac{\cancel{R}}{\cancel{L}} e^{-\frac{R}{L}t}\right)\right]$$
$$P_L(t) = \frac{V_B}{R} e^{-\frac{R}{L}t}\left(1 - e^{-\frac{R}{L}t}\right)$$

Now we can find the power at $t = 1\tau = \frac{L}{R}$

$$P_R(\tau) = \frac{{V_B}^2}{R}\left(1 - e^{-\frac{\cancel{R}\,\cancel{L}}{\cancel{L}\,\cancel{R}}}\right)^2$$
$$P_R(\tau) = \frac{21^2}{4}(1 - e^{-1})^2$$
$$P_R(\tau) = 44.05\ W$$

$$P_L(\tau) = \frac{{V_B}^2}{R} e^{-\frac{\cancel{R}\,\cancel{L}}{\cancel{L}\,\cancel{R}}}\left(1 - e^{-\frac{\cancel{R}\,\cancel{L}}{\cancel{L}\,\cancel{R}}}\right)$$
$$P_L(\tau) = \frac{21^2}{4} e^{-1}(1 - e^{-1})$$
$$P_L(\tau) = 25.64\ W$$

Part c.)

The total energy stored in the inductor at any time t is given as:

$$U_L(t) = \frac{1}{2}Li(t)^2$$
$$U_L(t) = \frac{1}{2}L\left[\frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right]^2$$

Therefore, the energy stored at $t = 5\tau = 5\frac{L}{R}$ is as follows:

$$U_L(5\tau) = \frac{1}{2}L\left[\frac{V}{R}\left(1 - e^{-\frac{R}{L}5\frac{L}{R}}\right)\right]^2$$
$$U_L(5\tau) = \frac{3}{2}\left[\frac{21}{4}(1 - e^{-5})\right]^2$$
$$U_L(5\tau) = 40.79\,J$$

To find the total amount of energy dissipated by the resistor we need to integrate $P_R(t)$ from $t = 0$ to $t = 5\tau$. With $P_R(t) = i(t)^2R$ from above we have the following integral, which is not trivial.

$$E_R = \int_0^{5\frac{L}{R}} \left(\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right)^2 R\,dt$$

However, using conservation of energy we also know that the total energy delivered by the battery is equal to the amount dissipated from the resistor plus the amount stored in the inductor.

$$E_B(t) = E_R(t) + U_L(t)$$

And since we already know $U_L(5\tau)$, we can instead find $E_B(5\tau)$ by integrating $P_B(t)$, which is a less difficult integral to evaluate.

$$E_B = \int_0^{5\frac{L}{R}} P_B(t)\,dt$$

$$E_B = V_B \int_0^{5\frac{L}{R}} i(t)\,dt$$

$$E_B = V_B \int_0^{5\frac{L}{R}} \frac{V_B}{R}\left(1 - e^{-\frac{R}{L}t}\right) dt$$

$$E_B = \frac{{V_B}^2}{R} \int_0^{5\frac{L}{R}} \left(1 - e^{-\frac{R}{L}t}\right) dt$$

$$E_B = \frac{{V_B}^2}{R}\left[\left(5\frac{L}{R} + \frac{L}{R}e^{-\frac{R}{L}\cdot 5\frac{L}{R}}\right) - \left(0 + \frac{L}{R}e^{-0}\right)\right]$$

$$E_B = \frac{{V_B}^2}{R}\left[5\frac{L}{R} + \frac{L}{R}e^{-5} - \frac{L}{R}\right]$$

$$E_B = \frac{L{V_B}^2}{R^2}\left[4 + e^{-5}\right]$$

$$E_B = \frac{3 \cdot 21^2}{4^2}\left[4 + e^{-5}\right]$$

$$E_B = 331.31\,J$$

Finally, the energy dissipated by the resistor is

$$E_R = E_B - U_L$$

$$E_R = 331.31 - 40.79$$

$$E_R = 290.5$$

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