Faraday's Law Introduction

In the last section we saw that a current produces a magnetic field. In this section we find the opposite is also true; that a magnetic field can be made to establish a current flow in a nearby conductor. The law that governs this phenomenon is referred to as *Faraday's Law of Induction*. Before introducing Faraday's law, we need to introduce the concept of *magnetic flux*.

When studying Gauss's law, we required a way to calculate the amount of electric field that passes through a surface. In that case we defined the electric flux as $\Phi_E = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$. The magnetic flux is a measure of the amount of magnetic field that passes through a surface and is similarly defined.

$$\Phi_B = \int_S \boldsymbol{B} \cdot \boldsymbol{dA}$$

As an example, the figure below shows a constant magnetic field pointing in the y direction with magnitude of $B = 0.5\ T$. The magnetic flux through a circular shaped surface oriented at an angle of 60° with respect to the $x - z$ plane is computed as follows.

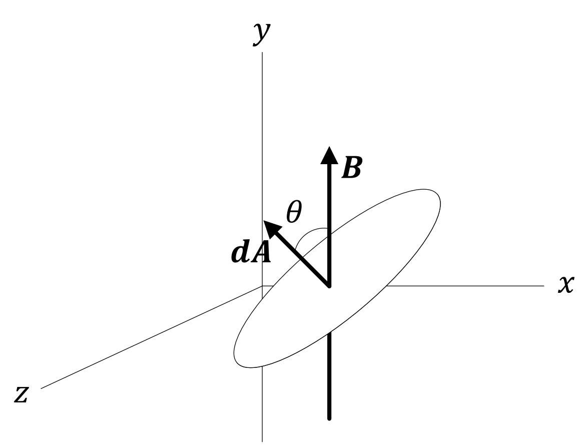
$$\Phi_B = \int_S \boldsymbol{B} \cdot \boldsymbol{dA}$$

$$\Phi_B = B\cos(\theta) \int_S \boldsymbol{dA}$$

$$\Phi_B = B\cos(\theta)\,\pi r^2$$

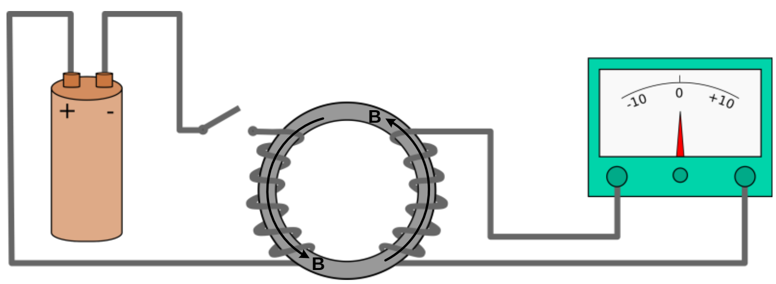
$$\Phi_B = 0.5 \cdot \cos(60) \cdot \pi \cdot 3^2$$

$$\Phi_B = 7.1\ T \cdot m^2$$



Faraday's Law of Induction

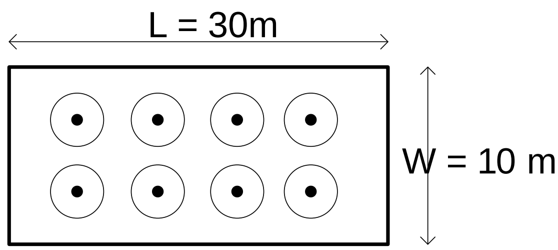
Once it was known that a current produces a magnetic field, scientist began to contemplate whether a magnetic field can also produce a current. Michael Faraday was one of the scientists who investigated this possibility. One of the experiments he used was to wrap two separate wires around opposite sides of an iron ring as shown below. One wire he connected to a battery and switch, while the other he connected to a galvanometer. When the switch is closed current flows through the wire that is connected to the battery, which in turn creates a magnetic field as shown in the figure. The hope was that a current would be *induced* in the second wire and would be detected by the galvanometer. What he observed however was that the galvanometer would register a current only for a short period of time after the switch is closed or open. Faraday understood that the current takes a finite amount of time to fully turn on/off when the switch is closed/open, which causes the magnetic field to change during these so-called transient times. With this he suspected that a *changing* magnetic field is required to induce a current in a second wire. Further experiments allowed a mathematically rigorous formulation of this phenomenon which is stated below as *Faraday's Law of Induction*.



Faraday's Law of Induction
The EMF, ε, induced around a closed path is equal to *the negative* of the time rate of change of the magnetic flux through the surface enclosed by the path.
$$\varepsilon = -\frac{d\Phi_B}{dt}$$ Where, $\Phi_B = \oint \boldsymbol{B} \cdot \boldsymbol{dA}$

Before explaining the negative sign in Faraday's law let's do an example and start by finding just the magnitude of the induced current.

A magnetic field directed out of the page is increasing at a rate of $A\ T/s$; i.e. $B(t) = At$, where $A = 0.6$. A rectangular loop of conducting wire with a resistance of $R = 20\ \Omega$ is placed in this field perpendicularly oriented as shown below. Let's find the magnitude of the current induced in the wire?



We start by finding the magnetic flux through the surface enclosed by the wire.

$$\Phi_B = \oint \boldsymbol{B} \cdot \boldsymbol{dA}$$
$$\Phi_B(t) = B(t) \oint dA$$
$$\Phi_B(t) = AtLW$$

With this we can write Faraday's law using the magnitude only.

$$|\varepsilon| = \left|-\frac{d\Phi_B}{dt}\right|$$
$$|\varepsilon| = \left|-\frac{d(LWAt)}{dt}\right|$$
$$|\varepsilon| = LWA$$

The magnitude of the current is then

$$|I| = \frac{|\varepsilon|}{R}$$
$$|I| = \frac{LWA}{R}$$
$$|I| = \frac{30 \cdot 10 \cdot 0.6}{20}$$
$$|I| = 9\ Amps$$

To determine the direction of the induced current we use Lenz's law.

Lenz's Law
A current that is induced in a conductor by a changing magnetic flux will have a direction such that the magnetic flux created by the induced current opposes the initial changing magnetic flux that created it.

Using this law, we can determine the direction of the induced current in the example above. The magnetic field, and hence the magnetic flux, was increasing in strength in a direction pointing out of the page. The induced current then should produce a magnetic field that points into the page to oppose this increase. Using the right-hand rule, we see that the induced current will be in the clockwise direction.

Summary for Faraday's Law of Induction

Faraday's Law of Induction
The EMF, ε, induced around a closed path is equal to *the negative* of the time rate of change of the magnetic flux through the surface enclosed by the path. $$\varepsilon = -\frac{d\Phi_B}{dt}$$ Where, $\Phi_B = \oint \boldsymbol{B} \cdot \boldsymbol{dA}$
Lenz's Law
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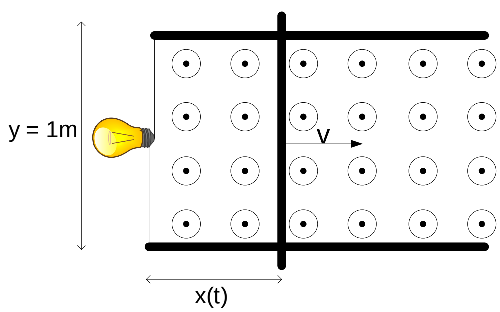
Examples:

Question 1:

A small bulb rated at $1.2\ V /\ 0.25\ W$ is connected to two frictionless metal rails, which are placed in a uniform magnetic field a with strength of $0.2\ T$ as shown below. A conducting rod is placed across the rails and a force is applied to the rod, so it begins to move to the right.

a.) What is the speed of the rod when the bulb fully lights up (i.e. applied voltage is $1.2\ V$)?
b.) What is the direction of the current?
c.) What is the external force required to keep the rod moving at a constant speed with the rate found in part a?



Solution 1:

Part a.)

As the conducting rod moves to the right the flux through the enclosed surface increases. The magnetic flux through the enclosed surface is given as:

$$\Phi_B = \oint \boldsymbol{B} \cdot \boldsymbol{dA}$$
$$\Phi_B = By \cdot x(t)$$

The magnitude of the EMF can then be found with Faraday's law.

$$|\varepsilon| = \left|\frac{d\Phi_B}{dt}\right|$$
$$|\varepsilon| = By\left|\frac{dx(t)}{dt}\right|$$
$$\varepsilon = Byv$$

To reach the required EMF, the velocity required is as given below.

$$v = \frac{\varepsilon}{By}$$
$$v = \frac{1.2}{0.2 \cdot 1} = 6\ m/s$$

Part b.)

Using Lenz's law, we find the direction of the current is clockwise so that the induced magnetic field is directed into the page to oppose the increasing magnetic flux.

Part c.)

From earlier lessons the force on a current carrying wire placed in a magnetic field is given by

$$\boldsymbol{F_B} = \boldsymbol{iLxB}$$

The figure below shows the magnetic force will be directed to the left with a magnitude of

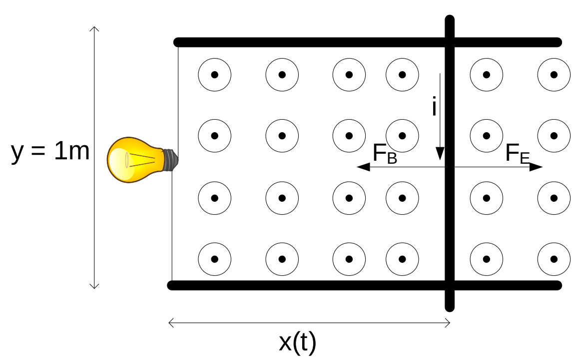
$$F_B = iLB$$

The current is related to the power and the voltage as $i = \frac{P}{\varepsilon}$, therefore we have.

$$F_B = \frac{P}{\varepsilon}LB$$

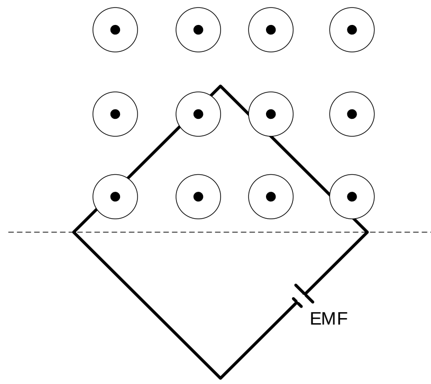
Finally, for the rod to move with constant velocity the magnitude of the external force must be equal to the magnitude of the magnetic force.

$$F_E = \frac{P}{\varepsilon}LB$$
$$F_E = \frac{0.25}{1.2} \cdot 1 \cdot 0.2$$
$$F_E = 0.042\ N$$



Question 2:

A square wire loop with $2.00\ m$ sides is perpendicular to a uniform magnetic field, with half the area of the loop in the field as shown below. The loop contains an ideal battery with $\varepsilon = 20V$. If the magnitude of the field varies with time according to $B = 0.0420 - 0.870t$, with B in teslas and t in seconds, what is the net EMF in the circuit and the direction of the net current around the loop?



Solution 2 :

The magnitude of the induced EMF is found using Faraday's law.

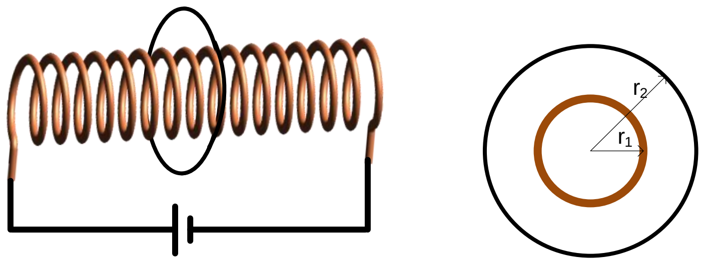
$$|\varepsilon| = \left|\frac{d\Phi_B}{dt}\right|$$
$$|\varepsilon| = \left|\frac{d\left(\frac{1}{2}s^2 \cdot (0.0420 - 0.870t)\right)}{dt}\right|$$
$$|\varepsilon| = \left|\frac{1}{2}2^2(0 - 0.870)\right|$$
$$|\varepsilon| = 1.74\ V$$

According to Lenz's law, since the magnetic field is pointing out of the page and decreasing in magnitude, the induced current will create a magnetic field to oppose this decrease and therefore also point out of the page. This results in a counter-clockwise current so that the induced EMF will add to the battery. The net EMF in the loop is then.

$$\varepsilon_{net} = 20 + 1.74$$
$$\varepsilon_{net} = 21.74$$

Question 3:

A ring of radius $r_2 = 10\ cm$ and a resistance of $0.0005\ \Omega$ is placed around the center of a long solenoid with $1400\ turns/m$ and a smaller radius of $r_1 = 4\ cm$. If the current in the solenoid is increasing at a rate of $280\ A/s$, what is magnitude of the induced current in the ring?



Solution 3:

From the previous section we know that the magnitude of the magnetic field inside a solenoid is given by: $B = \mu_0 ni$. In this case the current is a function of time.

$$B(t) = \mu_0 n i(t)$$

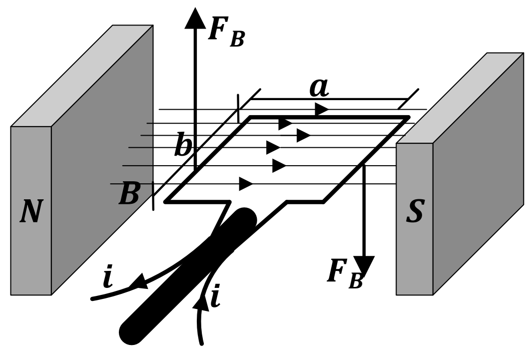
To find the magnitude of the current induced in the outer ring we again use Faraday's law, with $\varepsilon = iR$.

$$|iR| = \left|\frac{d\Phi_B}{dt}\right|$$
$$|i| = \frac{1}{R}\left|(\pi {r_1}^2 \mu_0 n)\frac{d}{dt}\big(i(t)\big)\right|$$
$$|i| = \frac{1}{R}|\pi {r_1}^2 \mu_0 n \cdot 280|$$
$$|i| = \frac{\pi \cdot 0.04^2 \cdot (4\pi E^{-7}) \cdot 1400 \cdot 280}{0.0005}$$
$$|i| = 4.95\ Amps$$

Note: We used the smaller radius of the solenoid since we assume the magnetic field outside the solenoid is negligible.

Question 4: The Electric Generator

Recall from an earlier lesson when we placed a current carrying rigid loop in a uniform magnetic field a torque was established. In our analysis we saw that if we used an alternating current, (one that switches the direction of the current in a periodic fashion), we can achieve a constant rotation. If the rotating loop is connected to a shaft, we can use this to do work (e.g. turn a fan). This is the basic principle of the electric motor.

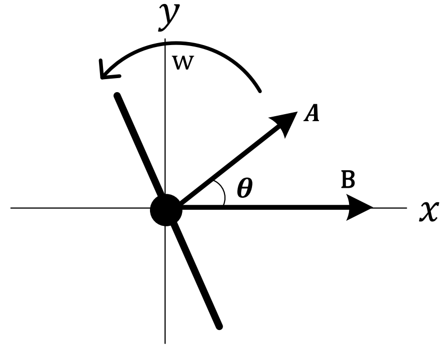


With our newfound knowledge of Faraday's law, it follows that if we instead *manually* rotated this rigid loop a current would be induced in the loop. This is the basic principle of the electric motor.

Assume the loop in the figure above has $N = 200$ turns with side lengths of $a = 0.05\ m, b = 0.06\ m$, and the magnetic field strength is $B = 0.1\ T$. We manually rotate this loop with a radial frequency of $\omega = 2\pi \cdot 60\ \ rad/s$. If the loop of wire is connected to a lightbulb with a resistance $6\ \Omega$, what is the magnitude of the current induced on the loop?

Solution 4:

The figure below shows our wire loop head-on and being rotated in a counter-clockwise manner.



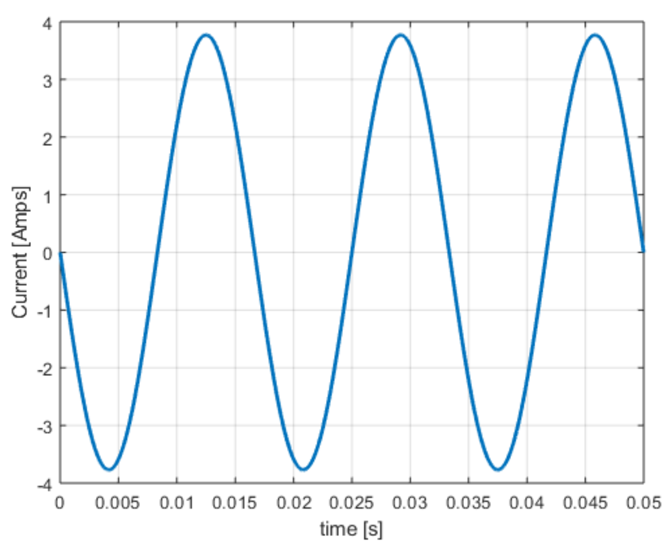
The magnetic flux through the loop is given as:

$$\Phi_B(t) = N(\boldsymbol{A} \cdot \boldsymbol{B})$$
$$\Phi_B(t) = N(AB\cos(\theta))$$
$$\Phi_B(t) = N(abB\cos(\omega t))$$

Using Faraday's law, the induced current is then

$$i(t) = \frac{1}{R}\frac{d}{dt}\{NabB\cos(\omega t)\}$$
$$i(t) = \left(\frac{NabB}{R}\right)\frac{d}{dt}\{\cos(\omega t)\}$$
$$i(t) = \left(\frac{NabB}{R}\right)(-\omega\sin(\omega t))$$
$$i(t) = \frac{-NabB\omega\sin(\omega t)}{R}$$
$$i(t) = \frac{-200 \cdot 0.05 \ \cdot 0.06 \cdot 0.1 \cdot 120\pi\sin(120\pi t)}{6}$$
$$i(t) = -3.77\sin(120\pi t)\ Amps$$

As an illustration the figure below shows three cycles of the induced current.



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