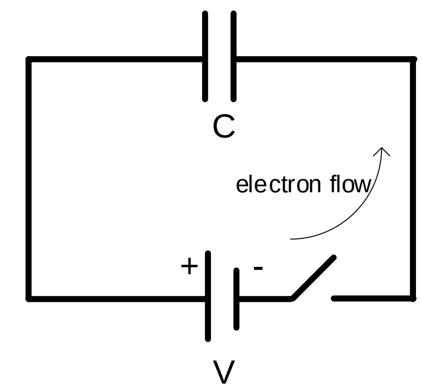
Resistor-Capacitor (RC) Circuits Introduction

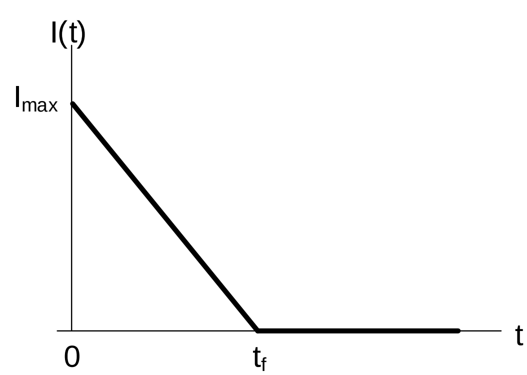
Although our primary study of electric current was with resistor circuits, it was with our study of capacitors where we first mentioned charge flow, albeit for a short period of time while the capacitor was being charged. We assumed this time was so small as to be inconsequential. The intention of this section, however, is to examine this so called “charging time”. Let’s look again at a simple capacitor circuit.



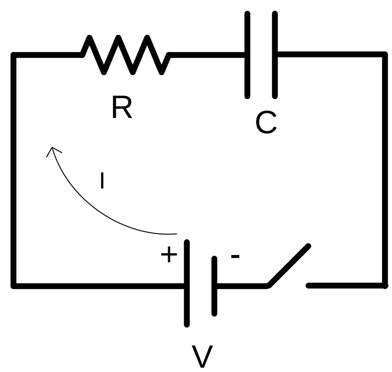
When we close the switch, electrons begin to accumulate on the plate of the capacitor. For illustrative purposes we can imagine the scenario as follows:

- As the plates are initially assumed neutral the first electron can race to the plate unimpeded.
- The next electron also races to the plate, but at a slower rate since it is now opposed by the first electron.
- The next electron, now opposed by the first *two* electrons races to the plate at an even slower rate.
- This “slowing” process continues for each subsequent electron until finally an electron has an insurmountable opposition force and the charge flow ends.

With this simple argument we can make a guess that the charge flow (electric current), during the charge time may look something like the figure below, where we close the switch at $t = 0$, and the capacitor becomes fully charged at $t = t_f$.



With our hypothesis made let's see if we can find a more exact mathematical description of the current flow during this "charging time". We re-draw the capacitor circuit below, except this time we place a resistor in series with the capacitor. Note even if we assume that a physical resistor is not present, we know that the wire itself (as well as the battery) has a certain amount of resistance, which is now accounted for. This circuit is formally referred to as an RC circuit.



To solve for the current flow, we start with Kirchhoff's voltage rule.

$$V - i(t)R - \frac{q(t)}{C} = 0$$

Note that since we are focused on the "charging time" we explicitly show that both the charge on the capacitor and the current flow are functions of time.

Next, we note that $i(t) = \frac{dq(t)}{dt}$ and write the following.

$$V - \frac{dq(t)}{dt}R - \frac{q(t)}{C} = 0$$

This is called a differential equation in q, as it involves not only the variable q, but the derivative of q. Various techniques can be used to solve this differential equation, but since usually one of the first techniques learned is called "separation of variables", we will use this technique. If you are not familiar with calculus you can skip the steps and proceed to the solution.

For the separation of variables technique, we separate the terms involving q and dq on one side and those involving t and dt of the other side.

$$V = \frac{dq}{dt}R + \frac{q}{C}$$

$$VCdt = RCdq + qdt$$

$$(VC - q)dt = RCdq$$

$$\frac{1}{(VC - q)}dq = \frac{1}{RC}dt$$

Now we can integrate both sides.

$$\int_0^q \frac{1}{(VC - q)}dq = \int_0^t \frac{1}{RC}dt$$

$$-\ln(VC - q)|_0^q = \frac{1}{RC}t$$

$$-(\ln(VC - q) - \ln(VC)) = \frac{1}{RC}t$$

$$\ln\left(1 - \frac{q}{VC}\right) = -\frac{1}{RC}t$$

$$\frac{VC - q}{VC} = e^{-\frac{1}{RC}t}$$

$$q(t) = VC\left(1 - e^{-\frac{1}{RC}t}\right)$$

And since $i(t) = \frac{dq(t)}{dt}$

$$i(t) = V\cancel{C}\frac{1}{R\cancel{C}}e^{-\frac{1}{RC}t}$$

$$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}$$

We have found that ehe electric current is described by a decaying exponential, very closely matching our intuition from above! At $t = 0$, we have the maximum value of the current, $\frac{V}{R}$, as would be expected since there is not yet any potential across the capacitor. Furthermore, as $t \to \infty$, the current goes to zero, again as expected. Let's now take a closer look at the constant term in the exponential. The denominator has units of time, as it should to cancel with the variable t.

$$RC = \frac{V}{I} \cdot \frac{Q}{V} = \frac{Q}{I} : \frac{Coulombs}{Coulombs/sec} = sec$$

This term will dictate the decay rate of the exponential and is called the *time constant*, τ.

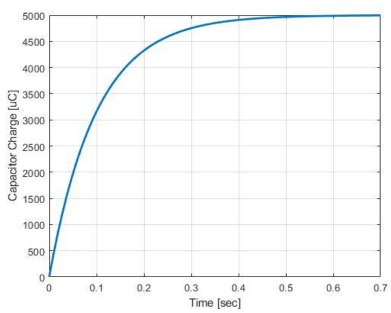
$$\tau = RC$$

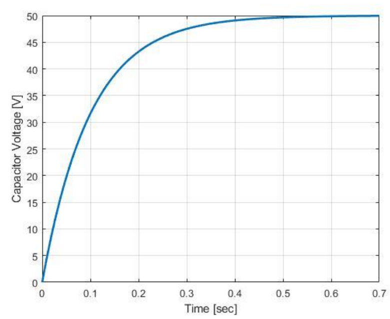
With respect to the current we can see that τ represents the time required for the current to drop to $\approx 37\%$ of its initial (maximum) value.

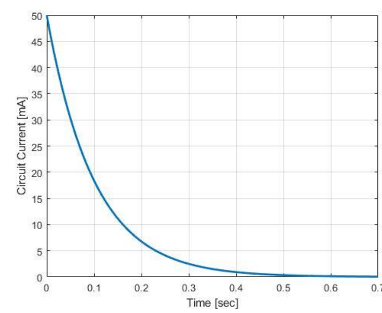
$$i(RC) = I_{max}e^{-\frac{1}{\cancel{RC}}\cancel{RC}}$$
$$i(RC) = I_{max}e^{-1} \approx 0.37I_{max}$$

Finally, using $V = \frac{Q}{C}$, we have the following three equations that represent the charge, current, and voltage across a capacitor for charging of an RC circuit. For illustrative purposes, we also show plots using $R = 1000\ \Omega$, and $C = 0.0001\ F$.

Charging a Capacitor in an RC Circuit		
Charge on Capacitor	*Voltage across Capacitor*	*Current in Circuit*
$q(t) = VC\left(1 - e^{-\frac{1}{RC}t}\right)$	$v(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$	$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}$

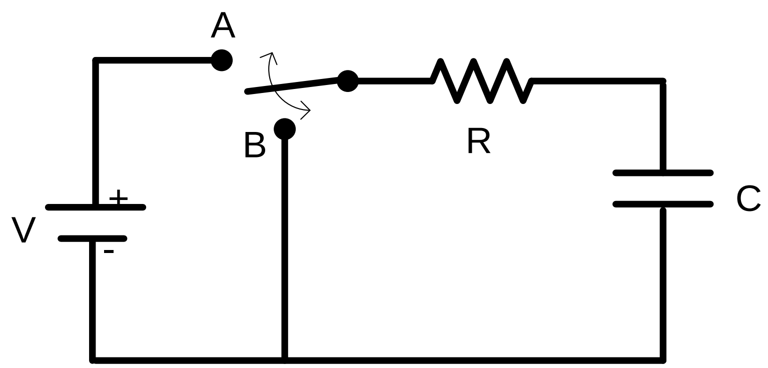






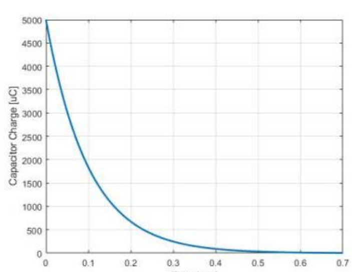
Capacitor Discharge:

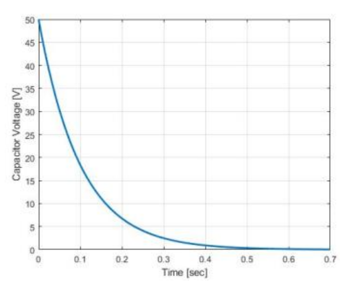
If after the capacitor is charged, we remove the voltage source and complete the circuit, the capacitor will discharge. In the circuit below let's assume the switch has been in position A for a long period of time such that the capacitor is fully charged. If we now move the switch to position B, the capacitor will begin to discharge.

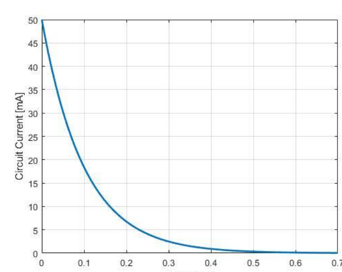


Before the switch is turned to position B the top plate of the capacitor is positively charged while the bottom is negatively charged. Once the switch is turned to position B electrons will begin to flow from the negative plate to the positive plate, until the plates again becomes neutral. As you could imagine the transient behavior of the charge, current, and voltages during discharge are similarly described by exponential functions. As the derivations closely follow what we have done for charging, we only show the results below.

Discharging a Capacitor in an RC Circuit		
Charge on Capacitor	*Voltage across Capacitor*	*Current in Circuit*
$q(t) = VCe^{-\frac{1}{RC}t}$	$v(t) = Ve^{-\frac{1}{RC}t}$	$i(t) = -\frac{V}{C}e^{-\frac{1}{RC}t}$







Final Summary for RC Circuits

Charging an RC Circuit

When a voltage source is applied to a resistor and capacitor in series the charge, current, and voltage vary according to the following equations.

Charge on Capacitor

$$q(t) = VC\left(1 - e^{-\frac{1}{RC}t}\right)$$

Voltage across Capacitor

$$v(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$$

Current in Circuit

$$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}$$

Discharging an RC Circuit

When a charged capacitor is put in series with a resistor and a completed circuit is created the charge, current, and voltage vary according to the following equations.

Charge on Capacitor

$$q(t) = VCe^{-\frac{1}{RC}t}$$

Voltage across Capacitor

$$v(t) = Ve^{-\frac{1}{RC}t}$$

Current in Circuit

$$i(t) = -\frac{V}{C}e^{-\frac{1}{RC}t}$$

RC Circuit Time Constant

The time constant, τ, for an RC circuit describes the transient behavior is the following ways.

- Charging
  - At $t = \tau$
    - The capacitor is $\approx 63\%$ charged.
    - The voltage across the capacitor is $\approx 63\%$ of its final value.
    - The current in the circuit is $\approx 37\%$ of its initial value.
- Discharging
  - At $t = \tau$
    - The capacitor has $\approx 37\%$ of its initial charge.
    - The voltage across the capacitor is $\approx 37\%$ of its final initial value.
    - The current in the circuit is $\approx 37\%$ of its initial value.

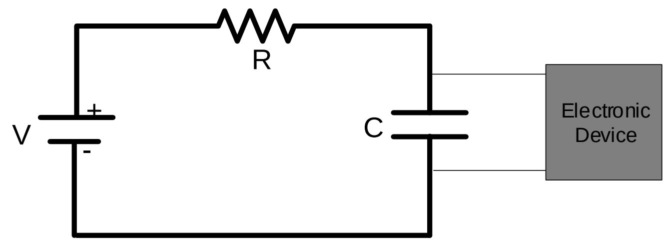
Where,

$$\tau = RC$$

Examples:

Question 1:

The circuit diagram below shows one way an RC circuit can be used to protect against power outages.



Choose an appropriate resistor for the circuit, with $C = 14\ uF$, if the electronic device requirement is that at least 70% of the supply voltage needs to be maintained for $0.20\ sec$ after a power outage.

Solution 1:

Assuming the capacitor is fully charged, the voltage supplied to the device is V before the power goes out. When the source voltage turns off (short circuit across the source), the capacitor begins to discharge through the resistor. Therefore, the voltage being supplied to the electronic device decays according to the below equation.

$$v(t) = Ve^{-\frac{1}{RC}t}$$

We need this voltage to maintain 70% of its value for $0.2\ sec$.

$$v(0.2) = \cancel{V}e^{-\frac{0.2}{RC}} = 0.7\cancel{V}$$

$$e^{-\frac{0.2}{RC}} = 0.7$$

We can solve for R by taking the natural logarithm of both sides.

$$\ln\left(e^{-\frac{0.2}{RC}}\right) = \ln 0.7$$

$$\frac{0.2}{RC} = -\ln 0.7$$

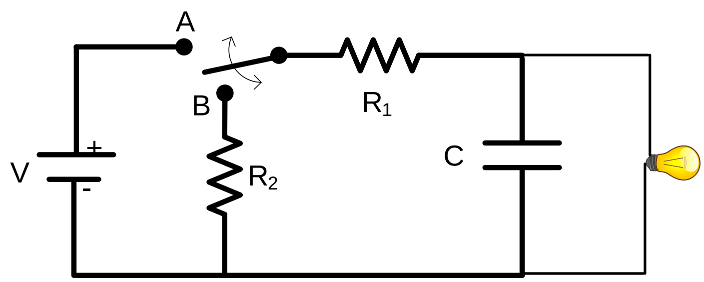
$$R = \frac{0.2}{-\ln(0.7)\,C}$$

$$R = \frac{0.2}{-\ln(0.7)\,14E^{-6}}$$

$$R = 40\ k\Omega$$

Question 2:

Charging and discharging of a capacitor can be used to slowly turn a light on and off. The circuit below can be used to illustrate this concept. The lightbulb is assumed to just start to glow when 3 volts is applied and reaches its maximum brightness at 10 volts. a.) Find the values of R_1 so that the light obtains full brightness 1.5 seconds after turning the switch to position A. b.) Assuming the light has been on for a long time, find the value of R_2 so that the light turns completely off 3 seconds after turning the switch to position B. The lightbulb can be assumed very high resistance so that its current draw is negligible, the voltage source is $12\ V$, and the capacitor value is $200\ uF$.



Solution 2:

When the switch is in position A we have a simple RC circuit, where $R = R_1$. The voltage across the capacitor, (and the bulb), is given by:

$$v(t) = V\left(1 - e^{-\frac{1}{R_1 C}t}\right)$$

Algebraically rearranging for R_1.

$$\frac{v(t)}{V} = 1 - e^{-\frac{1}{R_1 C}t}$$

$$\frac{t}{R_1 C} = -\ln\left(1 - \frac{v(t)}{V}\right)$$

$$R_1 = \frac{t}{-C \cdot \ln\left(1 - \frac{v(t)}{V}\right)}$$

The resistor, R_1, can be determined using the requirement that $v(t) = 10\ @\ t = 1.5$

$$R_1 = \frac{1.5}{-200E^{-6} \cdot \ln\left(1 - \frac{10}{12}\right)}$$

$$R_1 = 4816\ \Omega$$

When the switch is in position B, the capacitor will discharge through the series combination of R_1 and R_2.

$$v(t) = Ve^{-\frac{1}{(R_1+R_2)C}t}$$

Solving this equation for R_2 we have:

$$R_2 = \frac{t}{-C \cdot \ln\left(\frac{v(t)}{V}\right)} - R_1$$

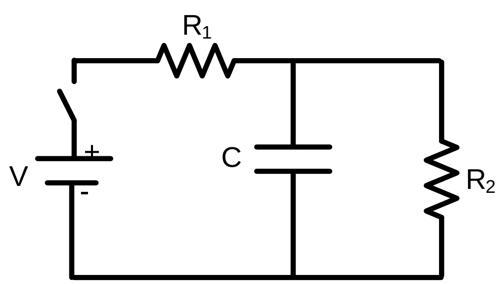
This time R_2, can be determined using the requirement that $v(t) = 3 \ @ \ t = 3.0$

$$R_2 = \frac{3}{-200E^{-6} \cdot \ln\left(\frac{3}{12}\right)} - 4816$$

$$R_1 = 6634\ \Omega$$

Question 3:

In the circuit below the switch has been open for a long time. a) In the instant the switch is closed what is the current supplied by the battery? b) What is the current a long time after the switch has been closed?



$$R_1 = 100\ \Omega$$
$$R_2 = 50\ \Omega$$
$$V = 20\ V\ \Omega$$

Solution 3:

In the first instant when the switch is closed the capacitor offers zero resistance (i.e. it acts like a short circuit), so that no current flows through the resistor, R_2. This results in the circuit consisting of the R_1 only, from which the current is found as follows.

$$I(t = 0) = \frac{V}{R_1} = \frac{20}{100} = 0.2\ A$$

After a long period of time with the switch closed the capacitor is fully charged and now offers infinite resistance (i.e. it acts like an open circuit), so that current flows only around the outer loop. This now results in the circuit consisting of the two resistors in series from which the current is found as shown.

$$I(t = 0) = \frac{V}{R_1 + R_2} = \frac{20}{150} = 0.13\ A$$

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