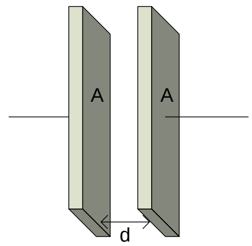
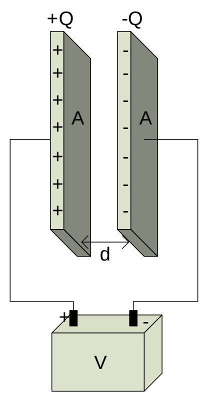
Capacitance Introduction

A *capacitor* is a device that can be used to store electric charge, and is commonly used in electronic circuits. In this context it generally consists of two conducting objects placed very near each other without physically touching. One of the simplest types of capacitor consists of two metal plates oriented parallel to each other as shown below.



The plates can be charged by applying an electric potential (voltage), for example from a battery, to the capacitor as shown here.



Once the battery is connected electrons will begin to flow away from the left plate towards the positive terminal of the battery, while at the same time electrons will flow from the negative terminal of the battery to the right plate. After a short period of time the potential between the plates will be equal the potential of the battery and the electrons will stop flowing. The plates of the capacitor will then be charged to $+Q$ and $-Q$ respectively. As you can imagine the larger the battery voltage the more charge will build up on the capacitor plates. The amount of charge on the plates per unit of voltage is what we refer to as *capacitance*.

$$C = \frac{Q}{V}$$

The unit we use is called the farad, (F), in honor of Michael Faraday.

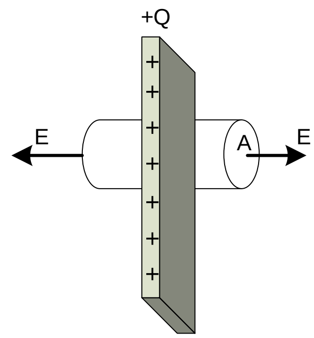
Solving the above equation for the charge we see directly that the higher the voltage the more charge that will deposit on the capacitor

$$Q = CV$$

Alternatively, if we could increase the capacitance, we could increase the amount of charge stored without needing to increase the voltage.

This is indeed possible and as it turns out the capacitance of a capacitor is a function of the geometry of the capacitor itself. Let's see if we can derive a purely geometrical relationship for the capacitance for the parallel plate capacitor.

We start by using Gauss's law to find the electric field due to a uniformly charged infinite conducting plate.



Using the cylinder as our Gaussian surface we write Gauss's law as follows:

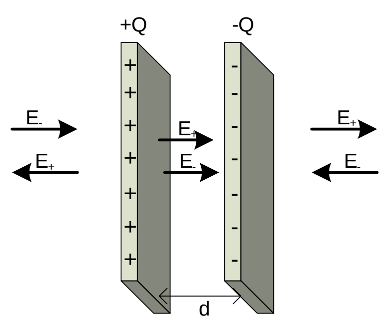
$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q}{\epsilon_0}$$

$$2EA = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{2A\epsilon_0}$$

Where we used the fact that the electric field is parallel to the face of the cylinder on both sides of the plate.

Putting two of these plates close to one another as we do for a capacitor we can see that the electric fields will combine between the plates and cancel outside the plates.



Therefore, the magnitude of the electric field between the plates of a parallel plate capacitor is twice the magnitude as is from a single plate.

$$E = \frac{Q}{A\epsilon_0}$$

Next, we recall from the previous section that for a uniform electric field the magnitude of the potential between two points parallel to the field is given as follows:

$$\Delta V = Ed$$

Solving for E and equating this expression with the expression for the electric field we found using Gauss's law we have.

$$\frac{\Delta V}{d} = \frac{Q}{A\epsilon_0}$$

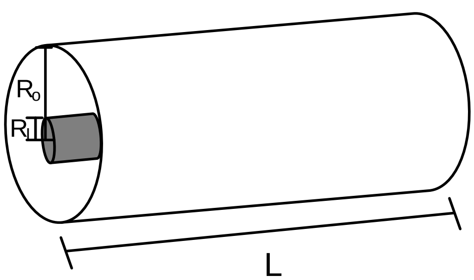
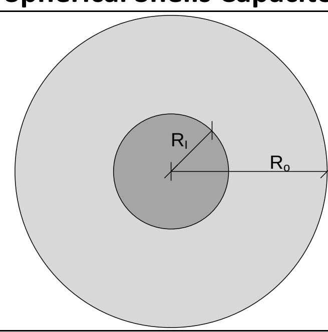
Finally, by solving for $\frac{Q}{\Delta V}$, which is what we called the capacitance, we have found a purely geometrical relationship for the capacitance of a parallel plate capacitor!

$$\frac{Q}{\Delta V} = \frac{A\epsilon_0}{d}$$

$$C = \epsilon_0 \frac{A}{d}$$

The equation shows that the amount of charge on a capacitor can be increased, while applying the same voltage, by either increasing the area or decreasing the distance between the plates.

Many other forms of capacitors are possible, of which two common forms are cylindrical and spherical. Geometrical expressions for their capacitance can similarly be derived, the results of which are provided below.

Cylindrical Capacitor (Coaxial Cable)	**Spherical Shells Capacitor**
	
$C = 2\pi\epsilon_0 \left(\frac{L}{\ln(R_O/R_I)}\right)$	$C = 4\pi\epsilon_0 \left(\frac{R_I}{1 - R_I/R_O}\right)$

Dielectric:

Note that the capacitors discussed above had air between the two conducting surfaces. If instead we place a sheet of insulating material, (such as paper or plastic), we find that the capacitance is increases by a factor of K. The material is referred to as a dielectric and K the dielectric constant.

$$C = KC_0$$

Where, C_0 is the capacitance when the space between the two conductors is filled with air. Different insulating materials have different values of K.

Electrical Energy Storage:

Once a capacitor is charged there exists an electrical potential between the two conductors, and therefore the capacitor can be said to store electrical potential energy. The stored energy is equal to the work done to charge it. A capacitor is charged by taking positive charge from one of the plates and moving it to the other plate. As mentioned earlier, it takes time to charge a capacitor, and the potential across the plates of a capacitor at any time during the charging process can be written as follows:

$$V(q) = \frac{q}{C}$$

Furthermore, during this process, the work done to move each infinitesimal charge, dq, is:

$$dW = V(q)dq$$

Integrating we find the total work required to fully charge the capacitor to a charge of Q.

$$W = \int_0^Q V(q)dq$$

$$W = \int_0^Q \frac{q}{C}dq$$

$$W = \frac{1}{C}\left[\frac{1}{2}(Q^2 - 0^2)\right]$$

Therefore, the energy stored in a fully charged capacitor can be written as:

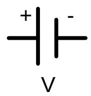
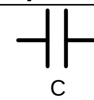
$$U = \frac{Q^2}{2C}$$

Using the relationship $C = \frac{Q}{V}$ we can derive two additional ways to express the stored energy, giving three fundamental relationships for the stored energy in a capacitor given below.

$$U = \frac{Q^2}{2C} \qquad U = \frac{CV^2}{2} \qquad U = \frac{QV}{2}$$

Capacitor Circuits:

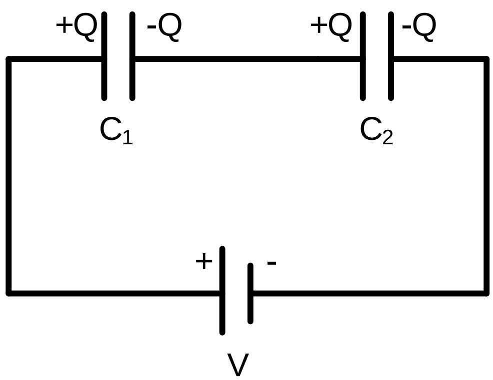
As mentioned, capacitors are commonly used on in electrical circuits. In general, electric circuits consists of a variety of different components, but the two we are now familiar with are the battery and the capacitor. To save time when drawing electrical circuits certain symbols have been adopted to represent various components, of which the battery and capacitor are shown below.

Battery	**Capacitor**
	

The two basic ways we can connect capacitors to each other in a circuit are in series or in parallel, which will be explained below.

Series Connections:

Below is an electrical circuit diagram showing two capacitors connected in series.



We can argue that the same amount of charge is on each capacitor in series by imagining the left plate of the first capacitor and the right plate of the second capacitor as the plates of a single capacitor. In this case we are already familiar with the fact that after a certain amount of time $+Q$ will appear on one plate while $-Q$ will appear on the other. Looking at the individual capacitors we can now argue that the $+Q$ on one plate of the first capacitor will induce a $-Q$ on the other plate, while the $-Q$ on one plate of the second capacitor will induce a $+Q$ on the other plate. Furthermore, the potential across both capacitors is equal to the battery voltage, and therefore it will be split between each capacitor. With these observations we can write the following equations to describe the above electrical circuit.

Voltage across Capacitor 1 with charge Q	**Voltage across Capacitor 2 with charge Q**	**Total voltage splits between the two capacitors**
$V_1 = \frac{Q}{C_1}$	$V_2 = \frac{Q}{C_2}$	$V = V_1 + V_2$

Substituting the first and second equation into the third we can find the equivalent capacitance of the two capacitors.

$$V = \frac{Q}{C_1} + \frac{Q}{C_2}$$
$$V = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$
$$V = \frac{Q}{C_{eq}}$$

Where:

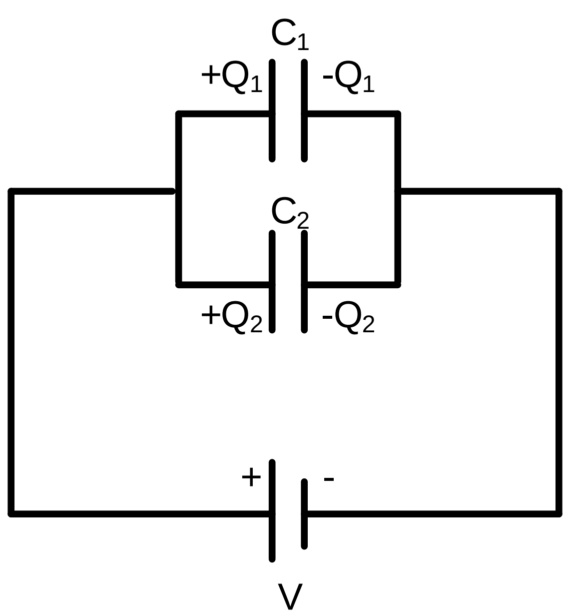
$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

Therefore, replacing the two capacitors from the above circuit with a single capacitor with a value of C_{eq}, will result in the same circuit behavior. The equation holds for any number of capacitors on series.

$$C_{eq,series} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)^{-1}$$

Parallel Connections:

Below is an electrical circuit diagram showing two capacitors connected in parallel.



In contrast to the series circuit, in this case we see that each capacitor maintains the same potential, but with different charges. For the above parallel circuit, the three fundamental equations then become as follows:

Charge on Capacitor 1 with voltage V	**Charge on Capacitor 2 with voltage V**	**Total charge splits between the two capacitors**
$Q_1 = C_1V$	$Q_2 = C_2V$	$Q = Q_1 + Q_2$

Like we did above we substitute the first and second equation into the third equation.

$$Q = C_1V + C_2V$$

$$Q = V(C_1 + C_2)$$

$$Q = VC_{eq}$$

Where:

$$C_{eq} = (C_1 + C_2)$$

In this case the equivalent capacitance for any number of capacitors in parallel is given as follows:

$$C_{eq,parallel} = (C_1 + C_2 + \cdots + C_N)$$

Final Summary for Capacitance

Capacitor and Capacitance
A *capacitor* is a device that can be used to store electric charge. It consists of two conducting objects placed near each other without physically touching. The capacitance, which can be measured experimentally, is a measure of the amount of charge that is stored per unit of potential applied. $$C \stackrel{\text{def}}{=} \frac{Q}{V}$$
Capacitor Types
The capacitance can also be expressed purely as a function of the geometry of the capacitor. *Parallel Plate Capacitor:* $$C = \epsilon_0 \frac{A}{d}$$ Where, A is the area of the plates and d is the distance between the plates. *Cylindrical Capacitor:* $$C = 2\pi\epsilon_0 \left(\frac{L}{\ln(R_O/R_I)}\right)$$ Where, L is the length of the cylinder, R_O is the radius of the outer cylinder, and R_I is the radius of the inner cylinder. *Spherical Capacitor:* $$C = 4\pi\epsilon_0 \left(\frac{R_I}{1 - R_I/R_O}\right)$$ Where, R_O is the radius of the outer sphere, and R_I is the radius of the inner sphere.
Energy Storage
The energy stored in a capacitor can be expressed as follows: $$U = \frac{Q^2}{2C} \qquad U = \frac{CV^2}{2} \qquad U = \frac{QV}{2}$$

Capacitor Circuits

Capacitors in Series:

When a voltage is applied across N capacitors in series, each capacitor will hold the same amount of charge. The equivalent capacitance for N capacitors connected in series will be less than any individual capacitor, and is given as follows:

$$C_{eq,series} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)^{-1}$$

Capacitors in Parallel:

When a voltage is applied across N capacitors in parallel, each capacitor will maintain the same voltage. The equivalent capacitance for N capacitors connected in parallel will be greater than any individual capacitor, and is given as follows:

$$C_{eq,parallel} = (C_1 + C_2 + \cdots + C_N)$$

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