Electric Potential Introduction – Part 3

In part 2 we learned how to find the electric potential if we know the electric field. We also mentioned that determining the potential of charge distributions is usually easier than finding the electric field since the former is a scalar function and the latter is a vector function. However, if we ultimately desire knowledge of the electric field is it possible to determine this once we know the electric potential? The answer is yes, and we will cover this topic below. We will also investigate the electric potential of charged conductors. If you recall, we have already argued that the electric field inside conductors is zero. The question in this section is; what can this tell us about the electric potential?

Electrical Field from the Electric Potential:

Recall the relationship we derived earlier to find the difference in the electric potential between two different points from the electric field.

$$\Delta V = -\int_i^f \boldsymbol{E} \cdot \boldsymbol{dl}$$

We can remove the integral in this relationship if we assume an infinitesimal potential difference, dV, between two points that are dl apart.

$$dV = -\boldsymbol{E} \cdot \boldsymbol{dl}$$

We can further let E_l be the component of the electric field in the direction of $\boldsymbol{dl}$ and write:

$$dV = -E_l dl$$

Finally, solving for the electric field we have the following.

$$E_l = -\frac{dV}{dl}$$

This gives us a way to find the electric field in a certain direction by computing the rate of change, (the derivative), of the potential in that same direction.

To generalize we assume the potential changes in three-dimensional space and we find the electric field separately in all three dimensions using the following three partial derivatives.

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

We can express these relationships more compactly using vector notation as follows:

$$\boldsymbol{E} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}$$

This expression is simply a way to compute the derivative of multivariable function. Note however that the result of this “derivative” is a vector valued function, while taking the derivative of single variable function results in a scalar function. A shorthand notation is generally used in mathematics for differentiation of multivariable functions using the *nabla* symbol, ∇, which we call the *del* operator.

$$\nabla \overset{\text{def}}{=} \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial x}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial x}\widehat{\boldsymbol{k}}$$

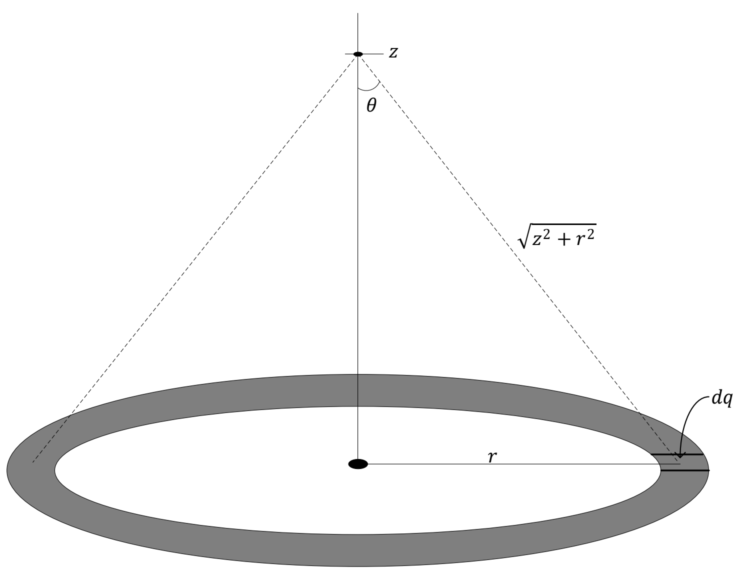
When this operator is applied to a scalar function, (i.e. the electric potential), the results is referred to as the gradient of the function operated on. Therefore, the electric field vector is equal to the negative gradient of the electric potential.

$$\boldsymbol{E} = -\nabla \mathrm{V}$$
$$\boldsymbol{E} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial x}\widehat{\boldsymbol{k}}$$

Let’s look at an example to see the possible benefits of our newly discovered relationship.

In our review of the electric field we derived an expression for the electric field from a charged disk. The derivation was not trivial and of course had to consider the vector nature of the electric field. Let’s take a different approach to this problem by first finding the electric potential from the same charged disk. Once we have an expression for the potential we can use the gradient operator to write an expression for the electric field.

We start the same way we did before with a charged ring.



Not having to be concerned with vectors as we did when working with the electric field the potential due to an infinitesimal charge on the ring is simply given by.

$$dV = k\frac{dq}{\sqrt{z^2 + r^2}}$$

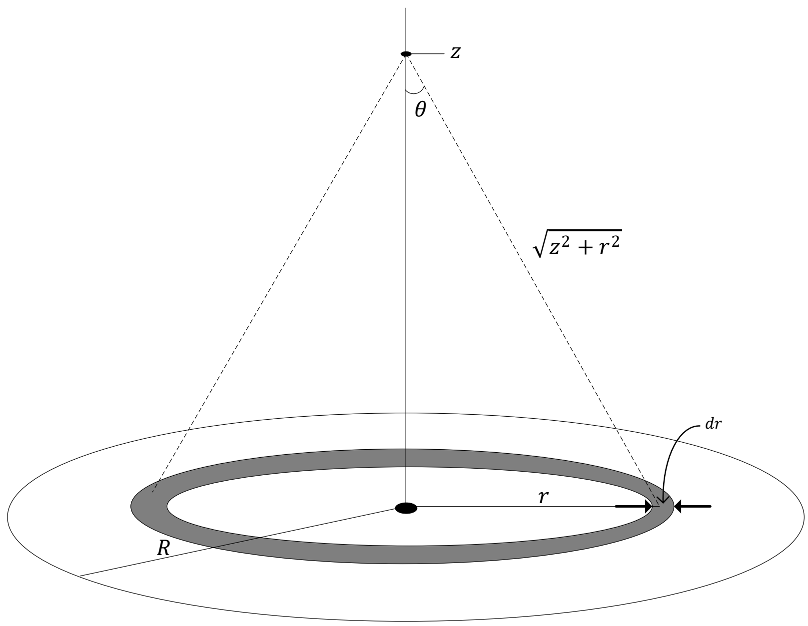
As we have done before we integrate to find the potential from all such sections.

$$V = k \int \frac{1}{\sqrt{z^2 + r^2}} dq$$

And since the distance is the same for all points around the ring this integral is quite trivial.

$$V = \frac{k}{\sqrt{z^2 + r^2}} \int dq$$
$$V = \frac{kQ}{\sqrt{z^2 + r^2}}$$

We now proceed as we did with the electric field example and construct a charged disk with radius R from infinitesimal concentric charged rings.



Based on our previous result we can express the potential from one of the infinitesimal rings at a distance r from the center as:

$$dV = \frac{kdq}{\sqrt{z^2 + r^2}}$$

If we assume the disk carries a uniformly distributed charge of Q, then the charge contained in a ring of thickness dr is proportional to the area of the ring.

$$\frac{dq}{Q} = \frac{2\cancel{\pi} rdr}{\cancel{\pi} R^2}$$
$$dq = \frac{2Qrdr}{R^2}$$

Substituting and integrating from 0 to R we have:

$$V = \int_0^R \frac{k2Qr}{R^2\sqrt{z^2+r^2}}dr$$

$$V = \frac{k2Q}{R^2}\int_0^R \frac{r}{\sqrt{z^2+r^2}}dr$$

This integral can be solved using substitution as follows:

$$u = z^2 + r^2$$
$$du = 2rdr$$

$$V = \frac{k2Q}{R^2}\int_{z^2}^{z^2+R^2} \frac{1}{2\sqrt{u}}du$$

$$V = \left(\frac{k\not{2}Q}{\not{2}R^2}\right)\left(2\sqrt{u}\right)_{z^2}^{z^2+R^2}$$

$$V = \left(\frac{k2Q}{R^2}\right)\left(\sqrt{z^2+R^2} - z\right)$$

Now that we have an expression for the electric potential, let's use the relationship we derived above to find the electric field.

$$\boldsymbol{E} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}$$

Where the first two components are zero since the electric potential is a function of z only, and we can write.

$$E_z = -\frac{dV}{dz}$$

$$E_z = -\left(\frac{k2Q}{R^2}\right)\frac{d}{dx}\left(\sqrt{z^2+R^2} - z\right)$$

$$E_z = -\left(\frac{k2Q}{R^2}\right)\left(\frac{\not{2}z}{\not{2}\sqrt{z^2+R^2}} - 1\right)$$

$$E_z = \left(\frac{2kQ}{R^2}\right)\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

Finally, replacing the total charge Q with the charge density, $\sigma = \frac{Q}{\pi R^2}$ we have:

$$E_z = \left(\frac{2k\sigma\pi \not{R^2}}{\not{R^2}}\right)\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$E_z = (2k\sigma\pi)\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

Which is the same expression we derived for the electric field in the previous section!

Electrical Potential of Charged Conductors:

When reviewing Gauss's law, we concluded that if you place a conductor in an electric field the resulting electric field anywhere inside the conductor is zero. We also used Gauss's law to show that if we place an excess charge on a conductor the charge will remain on the surface only. This was true even for a conductor with an empty cavity inside. We formally stated these results as shown below.

Theorem for Static Charges and Conductors
If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor. In other words, the electric field inside of a conductor is zero in the static situation.
Conducting Shells
A conducting shell with an excess charge will still have this excess charge move to the outside of the shell and will contain zero electric field inside the shell.

In this section we investigate the electric potential inside a conductor. To start let's write the definition for the change in the potential.

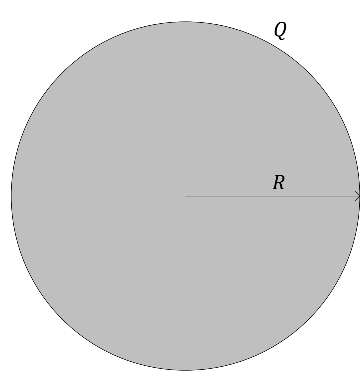
$$\Delta V = -\int_i^f \boldsymbol{E} \cdot \boldsymbol{dl}$$

And if the electric field is zero, as it is inside a conductor, we have the following:

$$V_f - V_i = -\int_i^f \boldsymbol{0} \cdot \boldsymbol{dl}$$
$$V_f = V_i$$

Which shows that the electric potential inside a conductor is constant. To understand how we can determine this constant value we will look at an example for a spherical conductor. Note that as the example shows a solid sphere, the same results will hold for a spherical shell since the electric field inside the shell is also zero.

The figure below shows a solid conducting sphere of radius R with a uniformly distributed surface charge of Q.



The magnitude of the electric field inside and outside the sphere can be found using Gauss's law, which we reviewed in a previous section.

$$E = \begin{cases} 0, & r < R \\ k\dfrac{Q}{r^2}, & r \geq R \end{cases}$$

To find the potential at a distance r from the center of the sphere we start with the equation from part 2 that relates the electric field to the change in the electric potential. If we assume the initial position at infinity and let $V(\infty) = 0$, we can solve the following integral in two sperate regions.

$$V(r) = -\int_{\infty}^{r} E dr$$

Outside the sphere, $r \geq R$:

$$V(r) = -\int_{\infty}^{r} E dr$$

$$V(r) = -kq \int_{\infty}^{r} \frac{1}{r^2} dr$$

$$V(r) = \left[k\frac{Q}{r} - k\frac{Q}{\infty}\right]$$

$$V(r) = k\frac{Q}{r}$$

Inside the sphere, $r < R$:

$$V(r) = -\int_{\infty}^{r} E dr$$

$$V(r) = \left(-\int_{\infty}^{R} E dr - \int_{R}^{r} E dr\right)$$

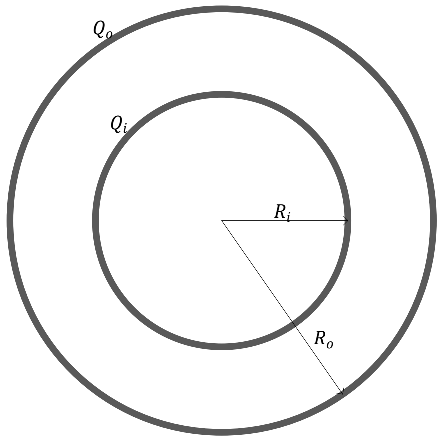
$$V(r) = \left(k\frac{Q}{R} - \int_{R}^{r} 0 dr\right)$$

$$V = k\frac{Q}{R}$$

Which shows that for all points inside the sphere the electric potential is constant and equal to the value on the surface of the sphere. Note again that this result also holds for a spherical shell.

Following this integral procedure for more complex examples can become time consuming. However, we can directly apply these results to more complex scenarios by using the superposition principle. Recall that when applying the principle of superposition, we sum the potentials from each charge *individually*. The following example illustrates this procedure.

Find the electric potential at a distance r from the center of two thin conducting concentric spherical shells shown below. The interior shell has a charge of $Q_i$and radius R_i, while the outer shell has a charge of Q_o and radius of R_o.



1. *Inside the inner shell,* $r < R_i$*:*

Using the principle of superposition, we can write:

$$V = V_{Q_i} + V_{Q_o}$$

Where, V_{Q_i} is the potential from the charge on the inner shell, and $V_{Q_o}$is the potential from the charge on the outer shell. From the first example we know that the potential inside a spherical shell is equal to the potential on the surface of the shell, therefore we have the following:

$$V = k\left(\frac{Q_i}{R_i} + \frac{Q_o}{R_o}\right)$$

2. *Between the two shells,* $R_i < r < R_o$*:*

We start with the same equation.

$$V = V_{Q_i} + V_{Q_o}$$

However, in this case since we are now *outside* the inner shell, the potential due to Q_i is not constant, but is a function of the distance.

$$V = k\left(\frac{Q_i}{r} + \frac{Q_o}{R_o}\right)$$

3. *Outside both shells,* $r > R_o$*:*

In this case we are outside both shells, therefore the potential from both shells are functions of the distance.

$$V = V_{Q_i} + V_{Q_o}$$
$$V = k\left(\frac{Q_i}{r} + \frac{Q_o}{r}\right)$$
$$V = k\left(\frac{Q_i + Q_o}{r}\right)$$

<u>Final Summary for Electric Potential</u>

Electric Potential from the Electric Potential
The electric field is equal to the negative gradient of the electric potential. $\boldsymbol{E} = -\nabla \mathrm{V}$ $\boldsymbol{E} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial x}\widehat{\boldsymbol{k}}$ Where each component of the electric field is equal to the partial derivative of the electric potential.
Electrical Potential of Charged Conductors
The electric potential inside a charged conductor is constant (i.e. all points within a charged conductor form an equipotential space.) The electric potential at points on the interior of a charged spherical conductor, (solid or shell), are equal to the electric potential at the surface.

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